STUDY OF AN ARRAY OF ELECTRICALLY SMALL RADIATORS

by

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DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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PL. MEIYAPPAN. B. E. (Hons)

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CERTIFICATE

This is to certify that the work presented in this thesis entitled "**STUDY OF AN ARRAY OF CLOSELY SPACED ELECTRICALLY SMALL RADIATORS**" has been carried out by *PL.MEIYAPPAN* under my supervision and the same has not been submitted elsewhere for a degree.

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ABSTRACT

In this thesis a new antenna configuration; an array of electrically small dipoles, is investigated as a possible solution to the requirement of a broad band antenna with a dipole like pattern. Multi octave bandwidth is what the proposed antenna configuration aims at. The configuration consists of a linear array of electrically small dipoles fed by a specially designed feed network,to maintain an input match as well as a dipole like current distribution over the entire band of frequencies.

The analysis is carried out in three parts. First, an individual electrically small dipole is analyzed to obtain its input characteristics as a function of frequency. In the second part mutual coupling between such dipoles kept close edge wise is analyzed. This resulted in characterizing the entire antenna array in terms of n-port scattering matrix,with each element of the matrix being a function of frequency. The third part consists of designing the terminal parameters of the feed network which provides the necessary driving point impedances and excitation voltages to the antenna element, which in turn maintains the desired current distribution on the antenna array. This analysis involves solution of simultaneous coupled equations which are solved using an iterative procedure. The best possible match at the feed port of the feed network as well as best match to the

desired driving point impedances at the antenna terminals, are obtained using an optimization algorithm based on a pattern search technique.

The analysis given in this thesis provides complete procedure for calculating optimum terminal parameters or in other words, the complete scatter matrix of the feed network as well as the antenna array. The actual design of the feed network is beyond the scope of this thesis.

CHAPTER -1 INTRODUCTION.

1.1 Introduction and literature survey :

Direction finding systems (DF) require omnidirectional pattern in azimuthal direction and preferably constant beam width in elevation over a wide band of frequency. These systems normally use resistive wire dipoles or biconical antennas for wide band coverage because a normal resonant $\lambda/2$ dipole has only 15% to 20% bandwidth. One of the drawbacks of resistive dipoles is that they are inefficient. The work reported in this thesis is an effort towards obtaining a highly efficient antenna while retaining the properties of wide band, omnidirectional coverage pattern etc., required for a DF system. In addition to these requirements desirable property of retaining the beam width constant in elevation over the band of frequencies of operation, has also been taken into consideration in this study.

Literature is abundant in the area of dipole analysis. Effort towards broad banding a dipole has been reported by several authors. Only the ones relevant to our study have been reviewed here. King and Wu [1] have shown that, dipoles will have wider bandwidth if the standing wave type of current distribution is converted into travelling wave type. This is due to the fact that, input impedance is a strong function of frequency for standing

wave current distribution, whereas travelling wave current makes it less dependant on frequency.

L.C.Shen [2] suggested a resistive loading technique to extend the usable bandwidth of dipoles. Obvious disadvantage with resistive loading technique is reduced efficiency. A study of reactive loading technique was carried out by B.L.Rao [3]. Most useful antenna, reported by Rao is an Exponentially Tapered Reactively Loaded Antenna [EXTRA]. This is reported to have 3:1 bandwidth.

B.J.Strait and K.Hiraawsa [4] have investigated multiple fed long wire antennas. They could achieve arbitrary high directivity in some direction by properly choosing the current distribution. This type of arbitrary current distribution needs optimized excitations and source impedances. Recently, Saudy[5] has employed the same technique in 0.5λ, 1λ, 1.5λ and 2λ dipoles to optimize gain and bandwidth. His analysis shows that an improvement is possible only at the expense of a complicated feed network and increased side lobe level.

1.2 Problem definition and proposed approach to the solution :

The aim of this study is to develop a radiator with dipole like pattern and having a multi octave bandwidth. The requirement in DF systems is that the coverage be omnidirectional in azimuth and preferably constant beamwidth in elevation. A typical $\lambda/2$ dipole pattern fits the pattern criteria very well except that dipole bandwidth is very small.

It is a well known that for good radiating efficiency the size of the antenna has to be at least comparable to quarter wavelength. A half wave dipole gives a good efficiency with about 15% to 20% bandwidth. Because of the resonant nature of the dipole fed at one point the bandwidth is limited. There have been studies in which a dipole is made to radiate efficiently at two or more frequency bands by inserting filter networks in the dipoles such that the dipole is resonant at different frequencies. Essentially these filter networks cut off the end sections at a higher frequency confining the current distribution to only the center sections. The work reported in this thesis is an extension of this idea in some way.

Our requirement here is to get a continuous broadband operation while maintaining dipole like pattern. Two conditions to be satisfied by the antenna become clear from this. One is that

the current distribution should be similar to a dipole current distribution at all frequencies in the operating band, which essentially means that we have to restrict the effective radiating length of the antenna to $\lambda/2$ at all the frequency. This also ensures high radiation efficiency for the antenna. The second condition is that the input should be matched to the source over the entire operating frequency band. This is sought to be achieved by the proposed antenna configuration.

The proposed configuration is that a linear antenna is built out of small dipoles, each with its feed point at the center of the dipole. The total length of the array of these dipoles is made equal to half wavelength at the lowest frequency of operation, so that it remains an efficient radiator over the entire band. Obviously, the individual dipole elements are electrically small at the lowest frequency, and would have very low efficiency. However, when all the elements taken together would produce a current distribution which resembles $\lambda/2$ resonant dipole current distribution, and would be efficient as an array. To obtain a match to the source over the frequency band of interest, a feed network with specially designed characteristics is inserted between ~~the~~ antenna elements and the source. This essentially would be a 1:n power divider network with the power distribution ratio of each port having a specially designed

function of frequency which in turn maintains the antenna current distribution to conform to a dipole like distribution with frequency. As frequency is increased from the lower end of the operating band, the extreme elements are progressively discon nected or the currents on these extreme elements are forced to zero, to keep the effective radiating length equal to $\lambda/2$. Having defined the configuration capable of giving the desired characteristics of dipole like pattern and broad bandwidth, the next step is to evolve a method to design such an antenna.

The configuration we have is a linear array of electrically small dipoles with close spacings and is fed by a feed network with some special characteristics. The problem is one of determining the length of the array, size of the elemental dipoles, number of dipoles, their spacings and the feed network parameters, given an operating frequency band. First two parameters can be easily arrived at without much analysis. From the current distribution requirement the total length of the array is fixed to be $\lambda/2$ at lowest operating frequency. At the highest operating frequency, it is envisaged that only one element will radiate, thus, the center element length is fixed to be $\lambda/2$ at the highest frequency. The same length is also taken to be element length. It is not necessary that this be so; one can have elements of different sizes making up the array. It is not yet determined

what is the best or optimum way of choosing the element lengths. In this study we have assumed identical elements for simplicity of analysis. Further spacing is assumed to be slightly longer than the element lengths so that t these elements are end to end, but do not touch each other. This gives nearly continuous current distribution over the array length. This, of course, imposes a condition that the array length be equal to integral multiples of $\lambda/2$ at the highest frequency. One can chose the highest and lowest frequencies such that this criteria is satisfied. Once element length and array length is determined, the number of elements is also automatically fixed. The number of elements is made odd so that we have a center element which alone radiates at the highest frequency of operation.

The analysis required to arrive at the design of the entire antenna system can be divided into several parts. Because the elements are electrically small and are closely spaced, the effect of mutual coupling is very strong, and it is important to characterize this accurately. The analysis is done in the following stages.

a) Analysis of electrically small individual, isolated radiator for its input impedance behavior over the frequency band of interest.

b) Analysis of mutual coupling between two such elements.

c) Input impedance behavior with frequency of the elements in an array.

d) Calculation of driving point impedances and excitation voltages required at each of the element inputs to sustain the desired dipole like current distribution.

e) Determination of feed network parameters which will provide the necessary excitation voltages and driving point impedances as a function of frequency

f) Finally, the design of the feed network itself from the required terminal parameters as arrived at step(e).

These steps (a) through (e) have been formulated and discussed in this thesis. The last step (f) has not been attempted.

1.3 Present investigations :

This work on array of electrically small radiators considers dipoles as individual radiators. To start with, electrically small dipole is analyzed in chapter-2. The input impedance is evaluated as a function of frequency, using well known Induced EMF method. Results of this chapter show that, electrically small radiators are highly reactive and are not efficient if operated as

an individual element antenna. The efficiency increases as the length of the element becomes comparable to resonant length $\lambda/2$.

Chapter-3 describes about mutual coupling aspects of closely spaced electrically small dipoles. Here also Induced EMF method is used to calculate mutual impedances. Then the equivalent circuit for the array is developed to evaluate driving point impedances and to study the effect of some termination on coupled antennas.

Next chapter is entirely devoted to feeding aspects of closely spaced array. Feed network parameters are determined such that a single source excites all the elements of array. Feed network synthesis uses an optimization routine to achieve minimum VSWR at the primary feed point.

This chapter is followed by a chapter on summary and conclusions. The synthesized S-matrix in conjugation with antenna elements are evaluated at many spot frequencies throughout the frequency of interest. Radiation pattern at different frequencies are also presented for the designed antenna. Some suggestions for further work are included at the end of the chapter.

CHAPTER 2. INPUT IMPEDANCE OF SMALL DIPOLES

2.1 Introduction :

An antenna can be represented by an equivalent impedance at the input terminals of the antenna, as far as the feed network is concerned. Impedance of an antenna at a single frequency can be written as:

$$Z_a = (R_r + R_l) + j X_a \qquad \ldots(2.1)$$

where,

R_r - Radiation resistance of the antenna

R_l - Loss resistance of the antenna

X_a - Reactance of the antenna

This impedance 'Z_a' at its terminals is called "Self impedance", which is the input impedance of the antenna in an isolated free space. In the presence of any interfering element (say some other antenna or electromagnetically sensitive material) the input impedance is modified by them and the impedance seen at its terminals is known to be "Driving Point Impedance". There are various methods to calculate the self impedance of the antenna. Generally these methods can be catogorised into

1. Boundary value method.

2. Equivalent Transmission Line method.

3. Poynting Vector Method.

4. Numerical Methods.

Out of these Poynting Vector Method [6] is the most popular one because of its simplicity. In the Poynting Vector Method, Power density is integrated, over a closed surface enclosing the antenna to yield the total power. Power divided by the square of the assumed current at the input terminals gives the input impedance.

2.2 Near fields of a dipole :

Referring to the figure 2.1 and assuming a sinusoidal current distribution along the element, as given by

$$I\,(x',y',z') = \hat{a}_z\, I_o \sin[\ k\,(l/2 - z')] \qquad 0 \le z' \le l/2$$

$$= \hat{a}_z\, Io \sin[\ k\,(l/2 + z')] \qquad -l/2 \le z' \le 0 \quad ; \qquad \ldots(2.2)$$

Vector potential '$\vec{A}$' can be written as,

$$\vec{A} = \hat{a}_z\, A_z$$

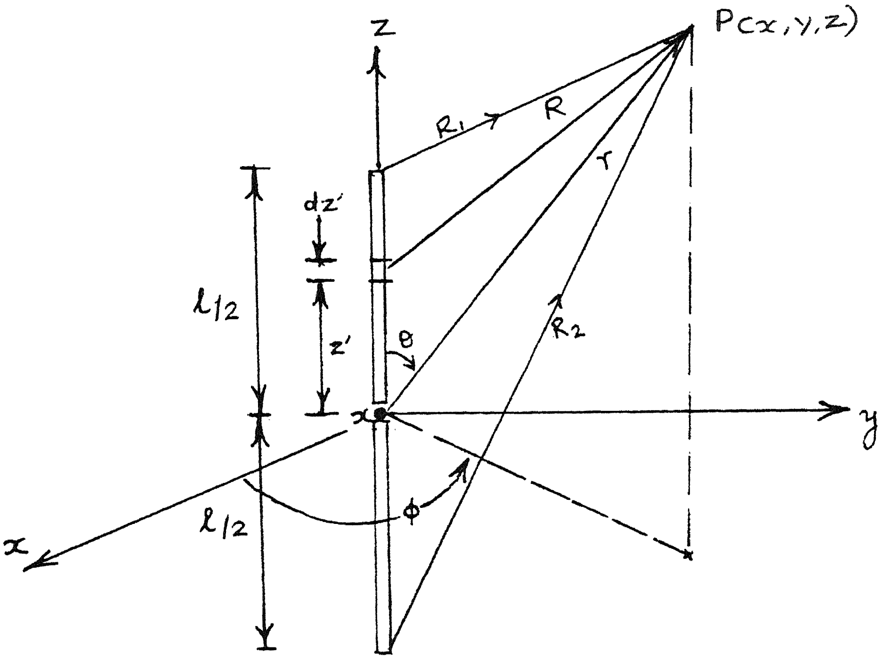


Fig 2.1 *Geometry of DIPOLE for near field calculations.*

$$\vec{A} = \hat{a}_z \frac{\mu_o I_o}{4\Pi} \left\{ \int_{-l/2}^{0} \sin[\,k\,(l/2 + z')] \frac{e^{-jkr}}{r}\, dz' \right.$$

$$\left. + \int_{0}^{l/2} \sin[\,k\,(l/2 - z')] \frac{e^{-jkr}}{r}\, dz' \right\} \quad \ldots(2.3)$$

If (X',Y',Z') are the source co-ordinates.

$$r = \sqrt{(x^2 + y^2 + (z-z')^2)}$$

$$= \sqrt{\rho^2 + (z-z')^2} \quad \ldots(2.4)$$

Magnetic field $\vec{H}$ is related to vector potential '$\vec{A}$' by

$$\vec{H} = 1/\mu \; \nabla \times \vec{A} \quad \ldots(2.5)$$

For an A_z component with no ϕ variations, in cylindrical co-ordinates, $\vec{H}$ becomes,

$$\vec{H} = - \frac{\hat{a}_\phi}{\mu} \frac{\partial A_z}{\partial \rho} \quad \ldots(2.6)$$

Since the field is not a function of ϕ, because of azimuthal symmetry, the observations can be made at any angle ϕ. Choosing phi is equal to $\pi/2$, x,y becomes,

$$X = \rho \cos\phi = 0 \; ;$$

$$Y = \rho \sin\phi = \rho \; ; \qquad \text{and } \partial/\partial\rho = \partial/\partial y.$$

$$\vec{H} = -\hat{a}_\phi \frac{I_o}{4\pi} \frac{\partial}{\partial y} \left\{ \int_{-l/2}^{0} \sin[\, k\,(l/2 + z')] \frac{e^{-jkr}}{r} dz' + \int_{0}^{l/2} \sin[\, k\,(l/2 - z')] \frac{e^{-jkr}}{r} dz' \right\} \qquad \ldots(2.7)$$

After some mathematical manipulation, $\vec{H}$ is simplified as,

$$\vec{H} = -\hat{a}_\phi \frac{I_o}{4\pi j} \frac{1}{y} \left\{ e^{-jkr_1} + e^{-jkr_2} - 2\cos(kl/2)\, e^{-jkr} \right\} \qquad \ldots(2.8)$$

Where R_1, R_2, R are shown in the fig. 2.1. Corresponding electric field components are given by, $\vec{E} = \nabla \times \vec{H} / j\omega\epsilon$ ... (2.9)

$$\vec{E}_\rho = \vec{E}_y = -\frac{1}{J\omega\varepsilon} \frac{\partial H_\phi}{\partial y} \qquad \ldots(2.10)$$

$$\vec{E}_z = \frac{1}{J\omega\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho H_\phi\right] = \frac{1}{J\omega\varepsilon}\frac{1}{y}\frac{\partial}{\partial y}\left[y H_\phi\right] \quad \ldots(2.11)$$

Substuting for H_ϕ

$$\vec{E}_\rho = \vec{E}_y = \frac{j\eta I_o}{4\pi y}\left\{\frac{e^{-jkr_1}}{r_1}\left[z - \frac{l}{2}\right] + \frac{e^{-jkr_2}}{r_2}\left[z + \frac{l}{2}\right] - 2Z\cos(kl/2)\frac{e^{-jkr}}{r}\right\} \quad \ldots(2.12)$$

and,

$$\vec{E}_z = -j\frac{\eta I_o}{4\pi}\left\{\frac{e^{-jkr_1}}{r_1} + \frac{e^{-jkr_2}}{r_2} - 2\cos(kl/2)\, e^{-jkr}\right\} \quad \ldots(2.13)$$

These are the required expressions for the fields radiated by a dipole.

2.3 Input Impedance by Induced EMF method :

Induced EMF method [7] can be briefly explained as follows. An EMF is applied at the terminals of the antenna, which produces a

current I_z . The current I_z produces an electric field E_z which in turn induces a field E_{zi} at the conductor surface such that the boundary conditions are satisfied. For a perfect conductor

$$E_{zt} = E_z + E_{zi} = 0 \qquad \text{at the surface}$$

Therefore $E_z = -E_{zi}$ ...(2.14)

Potential difference dV_z over an incremental length dz' is

$$dV_z = -E_z \, dz' \qquad ...(2.15)$$

This voltage is related to the current maximum by the transfer impedance i.e.,

$$Z_t = \frac{dV_z}{dI_m} \qquad ...(2.16)$$

Because of reciprocity, $Z_t = V_m / I_z$; where, V_m is the voltage at current maximum.

$$I_z \, dV_z = V_m dI_m \qquad ...(2.17)$$

Radiation impedance referred to current maximum is defined as,

$$Z_m = \frac{V_m}{I_m} = \frac{dV_m}{dI_m} \qquad ...(2.18)$$

From equation (2.17) and equation (2.18)

$$dV_m = \frac{I_z}{I_m} dV_z \qquad \text{...(2.19)}$$

$$V_m = -\int_{-l/2}^{l/2} dV_m = -\frac{1}{I_m} \int_{-l/2}^{l/2} I_z E_z \, dz' \qquad \text{...(2.20)}$$

Therefore,

$$Z_m = -\frac{1}{I_m^2} \int_{-l/2}^{l/2} I_z E_z \, dz' \qquad \text{...(2.21)}$$

where,

$$I_z = I_m \sin [\; k \; (l/2 - |z'| \;)] \qquad \text{...(2.22)}$$

and E_z is E field produced by the above current flowing in a filament placed along the axis of the wire, (derived in last section).

Substituting for E_z and I_z in equation (2.21) and manipulating, real and imaginary parts of impedance Z_m can be expressed as:

$$R_R = \frac{\eta}{2\pi} \Bigg\{ C + Ln\,(kl) - Ci\,(kl) +$$

$$1/2\ Sin\,(kl)\Big[\ Si\,(2kl) - 2\ Si(kl)\ \Big] +$$

$$1/2\ Cos\,(kl)\Big[\ C + ln\,(kl/2) + Ci\,(2kl) - 2\ Ci(kl)\ \Big]\Bigg\} \qquad \ldots(2.23)$$

and

$$X_s = \frac{\eta}{4\pi} \Bigg\{ 2\ Si\,(kl) +$$

$$Cos\,(kl)\Big[\ 2\ Si\,(kl) - Si(2kl)\ \Big]$$

$$-Sin\,(kl)\Big[\ 2\ Ci\,(kl) - Ci(2kl) - Ci(2ka^2/l)\ \Big]\Bigg\} \qquad \ldots(2.24)$$

where C = 0.5772 (Eulers' constant)

C_i and S_i are Cosine and Sine integrals given by:

$$C_i\,(X) = -\int_x^{\infty} \frac{Cos\ y}{y}\,dy = \int_{\infty}^{x} \frac{Cos\ y}{y}\,dy \qquad \ldots(2.25)$$

$$S_i\,(X) = \int_0^{x} \frac{Sin\ y}{y}\,dy \qquad \ldots(2.26)$$

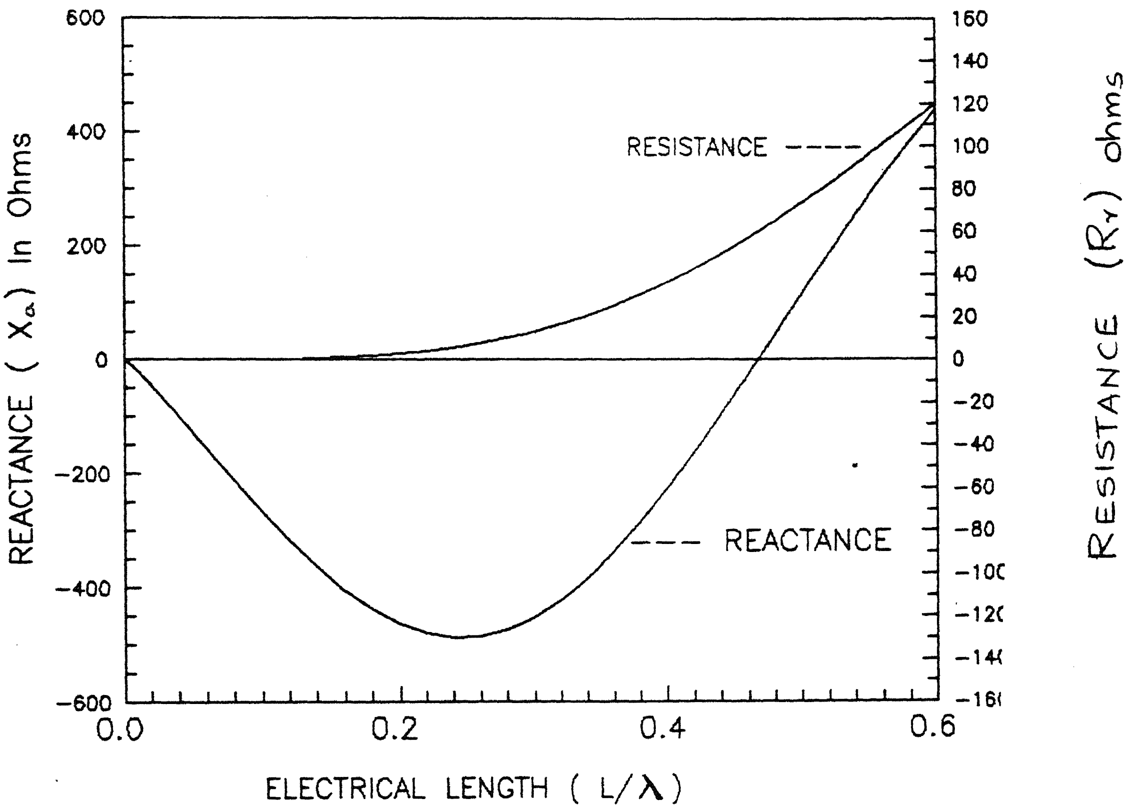


Fig 2.2 *Input reactance* (X_a) *and Radiation Resistance* (R_r) *of small DIPOLE.*

2.4 Results and Discussions:

A program 'Self.Pas' is written to calculate the input impedance of small dipoles, using equations (2.23) and (2.24). Fig (2.2) shows the plot of radiation resistance and input reactance as a function of length of dipole. It is observed that input impedance is dominated by the reactive part ($X_a \gg R_r$) and its slope is high. For (l/λ) less than 0.25 antenna is an inefficient radiator because $R_r \ll X_a$.

CHAPTER - 3

MUTUAL COUPLING BETWEEN CLOSELY SPACED ELECTRICALLY SMALL DIPOLES.

3.1 Introduction:

Antenna arrays are constructed by arranging N-individual elements in some specified configuration. In general, the behavior of each of these antennas in the presence of others will not be same as its isolated behavior. An antenna which is matched when isolated may no longer be so, when placed in an array and also the radiation pattern of an element in array will not in general be same as the radiation pattern of the element in isolated configuration.

These two properties arise due to the mutual coupling between the elements of the array. A study of mutual coupling between elements is essential to understand the behavior of the array. If the elements are placed close to each other, the effect of strong mutual coupling plays major role in array design.

3.2 Current Distribution In Coupled Dipoles:

Mutual coupling between the elements in a microwave network can be expressed in terms of S-parameters or Z-parameters. The advantage with the S-parameter is that it is easy to measure in practice and the Z-parameter is mostly used in theoretical computations. Mutual impedance (Z_{ij}) is the ratio of voltages at the i^{th} port due to the current in the j^{th} port and due to reciprocity, $Z_{ij} = Z_{ji}$.

The mutual impedance between two dipoles is calculated with the assumption that, current distribution in the first dipole is same as the current distribution in an isolated dipole. This is true only when the scattered fields due to the second antenna is zero, which is not possible. Hence to accurately find out the mutual coupling it is necessary to solve for the actual currents starting from boundary conditions and Maxwell's equations.

King et al [8] have given a rigorous formulation to find out the current distribution in coupled dipoles. the analysis leads to current as a function of position of antennas, given by a three term approximation.

$$i_{zm}(Z_m) = A_m M_{OZM} + B_M F_{OZM} + D H_{OZM} \qquad \ldots (3.1)$$

where A_m, B_m, D are complex constants dependant on the geometry. the functions M,F,H are given by,

$$M_{OZM} = \mathrm{Sin}\ \beta_o\ (\ h_i\ -\ |\ Z_i\ |\ ) \qquad \ldots(3.2.a)$$

$$F_{OZM} = \mathrm{Cos}\ \beta_o Z_m\ -\ \mathrm{Cos}\ \beta_o h_m \qquad \ldots(3.2.b)$$

$$H_{OZM} = \mathrm{Cos}\ \beta_o Z_m/2\ -\ \mathrm{Cos}\ \beta_o h_m/2 \qquad \ldots(3.2.c)$$

Where m is the primary or coupled antenna, Z_m represents the position along the length of the antenna 'm', h_m is the length of the antenna 'm'.

Evaluation of these three components individually, for various lengths of the antennas show that the contribution due to the last two terms F and H are small, in general, for $\lambda/2$ dipole and negligibly small for dipole lengths $<< \lambda/2$. Hence in the present analysis, the current distribution in the dipoles are assumed to be of first term (M_{ozm}) only.

3.3 Equivalent Circuit of Coupled Antennas:

As microwave engineers feel easy to visualize the terminal behaviour of the antenna in terms of equivalent circuit, it is desirable to represent coupled antennas also, by means of a multiport network. For simplicity two coupled antennae can be represented by a two port network.

The elements of T-equivalent circuit are represented in Z-parameters. The system of equations, characterizing it is,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad \ldots(3.3)$$

The input impedance or driving point impedance at port-1 is given by

$$Z_{IN1} = \frac{V_1}{I_1} = Z_{11} + Z_{12} \left[\frac{I_2}{I_1} \right] \qquad \ldots(3.4)$$

In a similar way n-element array can be represented by n-port equivalent circuit, characterized by means of self and mutual impedances. For an n-element array coupling matrix can be written as

$$[V] = [Z]\ [I] \qquad \ldots(3.5)$$

where

$$[V]^T = [V_1, V_2, V_3, \ldots, V_N]$$

$$[I]^T = [I_1, I_2, I_3, \ldots, I_N]$$

$$[z] = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & \cdots & Z_{1n} \\ Z_{21} & Z_{22} & Z_{23} & \cdots & Z_{2n} \\ Z_{31} & Z_{32} & Z_{33} & \cdots & Z_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ Z_{n1} & Z_{n2} & Z_{n3} & \cdots & Z_{nn} \end{bmatrix} \qquad \ldots(3.6)$$

Z_{ij} is the coupled or transfer or mutual impedance between ports i to j. Due to the reciprocity principle the coupling matrix will be symmetric about the main diagonal. This property is made use of in the design of feeding network in chapter 4.

The Z-parameters Z_{ii} ,Z_{ij} can be computed theoretically as described in the following section.

3.4 Mutual coupling in colinear configuration:

Mutual coupling in a linear planar array configuration has been analyzed extensively as early as 1948 [9]. Generalized treatment making use of induced EMF method explained in section 2.3 goes in the following manner.

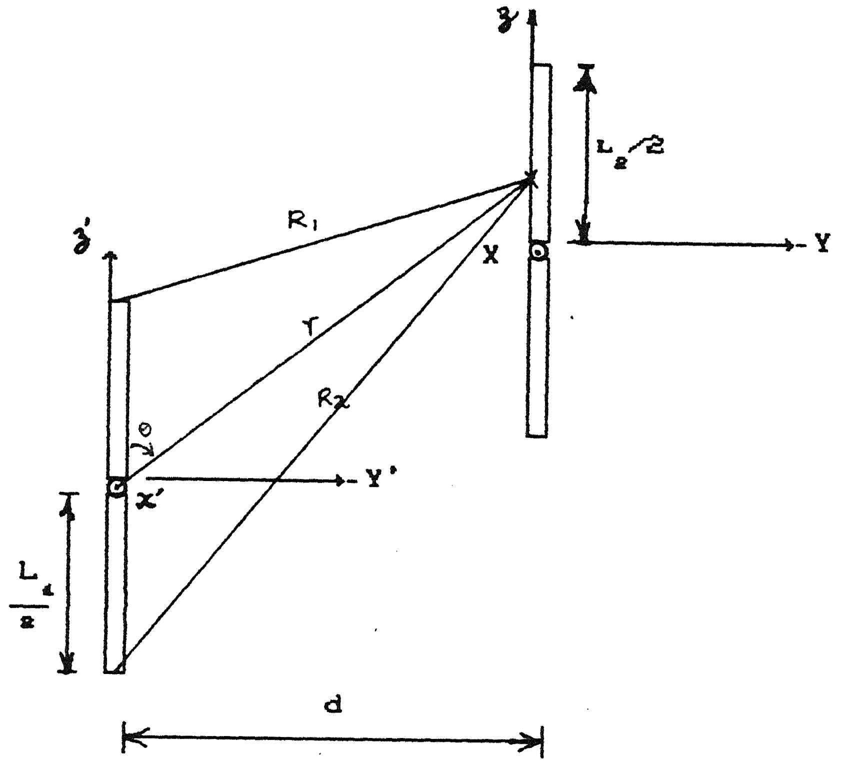


Fig 3.1. Generalised configuration for mutual coupling evaluation

The mutual impedance between the dipoles of fig (3.1) is defined by

$$Z_{21} = \frac{V_{21}}{I_1(0)} \qquad \ldots (3.7)$$

where V_{21} is the open circuit voltage at the terminals of antenna-2 due to $I_1(0)$, and $I_1(0)$ is the current at the center of antenna-1. The electric field due to the current $I_1(0)$ is known (derived in section 2.2 -Near fields of dipole). The parallel component E_{z21} along the axis of antenna-2 is calculated with the antenna-2 removed (temporarily). This gives the open circuit voltage across the terminals of antenna-2 as

$$V_{21} = - \frac{1}{I_1(0)} \int_{\text{antenna 2}} E_{z21} \, I_2 (z) \, dz \qquad \ldots (3.8)$$

So that the mutual impedance becomes,

$$Z_{21} = - \frac{1}{I_1(0) I_2(0)} \int_{\text{antenna 2}} E_{z21} \, I_2 (z) \, dz \qquad \ldots (3.9)$$

$I_2(z)$ is the current distribution along antenna-2 given by

$$I_2(z) = I_{2m} \operatorname{Sin} \beta \left[\frac{l_2}{2} - |z| \right] \qquad \ldots (3.10)$$

substituting this in the expression for E_{z21} equation (3.7) becomes

$$Z_{21} = \frac{j\,\eta}{4\,\pi} \int_{-l/2}^{+l/2} \operatorname{Sin}\left[k \frac{l_2}{2} - \left| Z \right| \right] \left\{ \frac{e^{-jkr_1}}{r_1} + \frac{e^{-jkr_2}}{r_2} - 2 \operatorname{Cos}(kl_1/2) \frac{e^{-jkr}}{r} \right\} dz \qquad \ldots (3.11)$$

For the colinear configuration, fig (3.2) real and imaginary parts of mutual impedances are expressed as

$$R_{12m} = - \frac{\eta}{8\pi} \left[\operatorname{Cos}(v_o) \left\{ -2 \operatorname{Ci}(2V_o) + \operatorname{Ci}(V2) + \operatorname{Ci}(v1) - \operatorname{Ln}(V3) \right\} \right] + \frac{\eta}{8\pi} \operatorname{Sin}(V_o) \left[2 \operatorname{Si}(2V_o) - \operatorname{Si}(V2) - \operatorname{Si}(V1) \right] \qquad \ldots (3.12)$$

$$X_{12m} = - \frac{\eta}{8\pi} \left[\operatorname{Cos}(v_o) \left\{ 2 \operatorname{Si}(2V_o) - \operatorname{Si}(V2) - \operatorname{Si}(v1) \right\} \right] + \frac{\eta}{8\pi} \operatorname{Sin}(V_o) \left[2 \operatorname{Ci}(2V_o) - \operatorname{Ci}(V2) - \operatorname{Ci}(V1) - \operatorname{Ln}(V3) \right] \qquad \ldots (3.13)$$

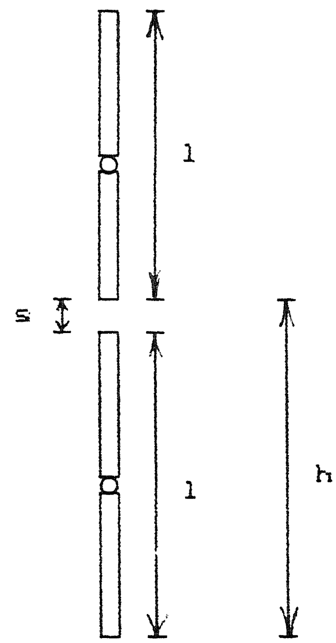


Fig 3.2. Colinear configuration.

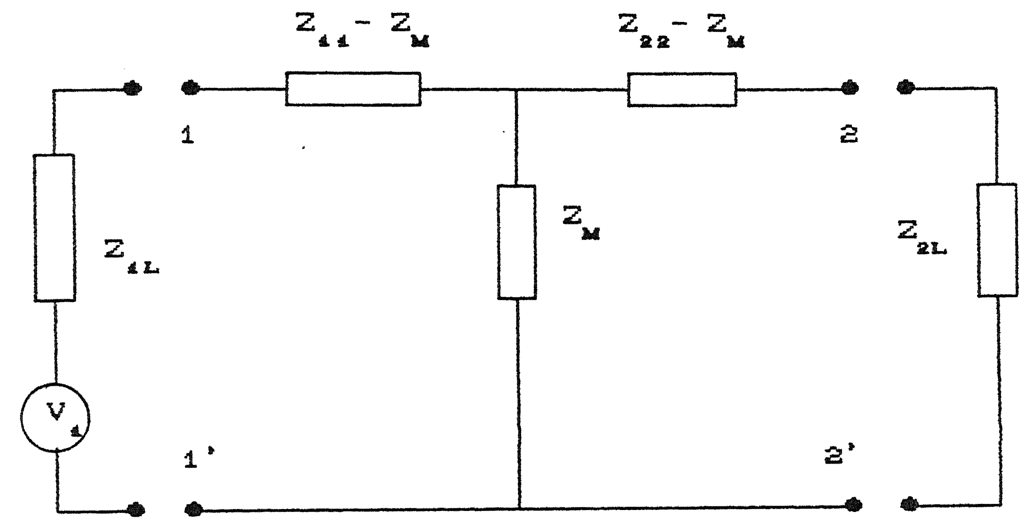


Fig 3.3 Eqivalent circuit of two element array with source and load

where $V_0 = Kh$

$V_1 = 2(K(h+1))$

$V_2 = 2(K(h-1))$

$V_3 = (h^2 - 1^2)/h^2$

Ci(x) and Si(x) are Cosine and Sine integrals given by equation (2.25) and equation (2.26).

3.5 Driving point impedance:

Equivalent circuit of two coupled antennae is described in section 3.3 and also in fig.3.3. This helps to visualize the effect of source and load impedances on the behaviour of the network. The driving point impedance of the coupled antennae should be matched to the source impedance at the respective terminals. For a two port network with source (or load) impedance Z_{1L} and Z_{2L} at port1 and port2, excitations V_1 and V_2 at port1 and port2 the driving point impedance can be calculated from the equivalent circuit as :

$$Z_{1d} = Z_{11} + Z_m (I_2 / I_1) \qquad \ldots (3.14)$$

$$Z_{d1} = Z_{11} + V_2 Z_m \left\{ \frac{Z_{11} + Z_{1L}}{V_1 \ (Z_{22} + Z_{2L}) - Z_M V_2} \right\} - \frac{Z_m^2}{Z_{22} + Z_{2L}}$$

$$- \frac{Z_M^3 \ V_2}{\left[Z_{22} + Z_{2L} \right] \left[(Z_{22} + Z_{2L}) V_1 - V_2 Z_M \right]}$$

...(3.15)

Driving point impedance at port 2 is given by a similar expression (interchanging subscripts 1 by 2). This says that driving point impedance at port1 is a function of load (or source impedance at the second port, and exciting voltages at port1 and port2, besides the mutual coupling impedance. Mutual impedance is fixed by the geometry of the array, but excitations and load or source impedances Z_{1L}, Z_{2L} are independent variables, which can be chosen to match the elements.

For an n-element array, characterized by the system of equations (3.5) driving point impedance is given by,

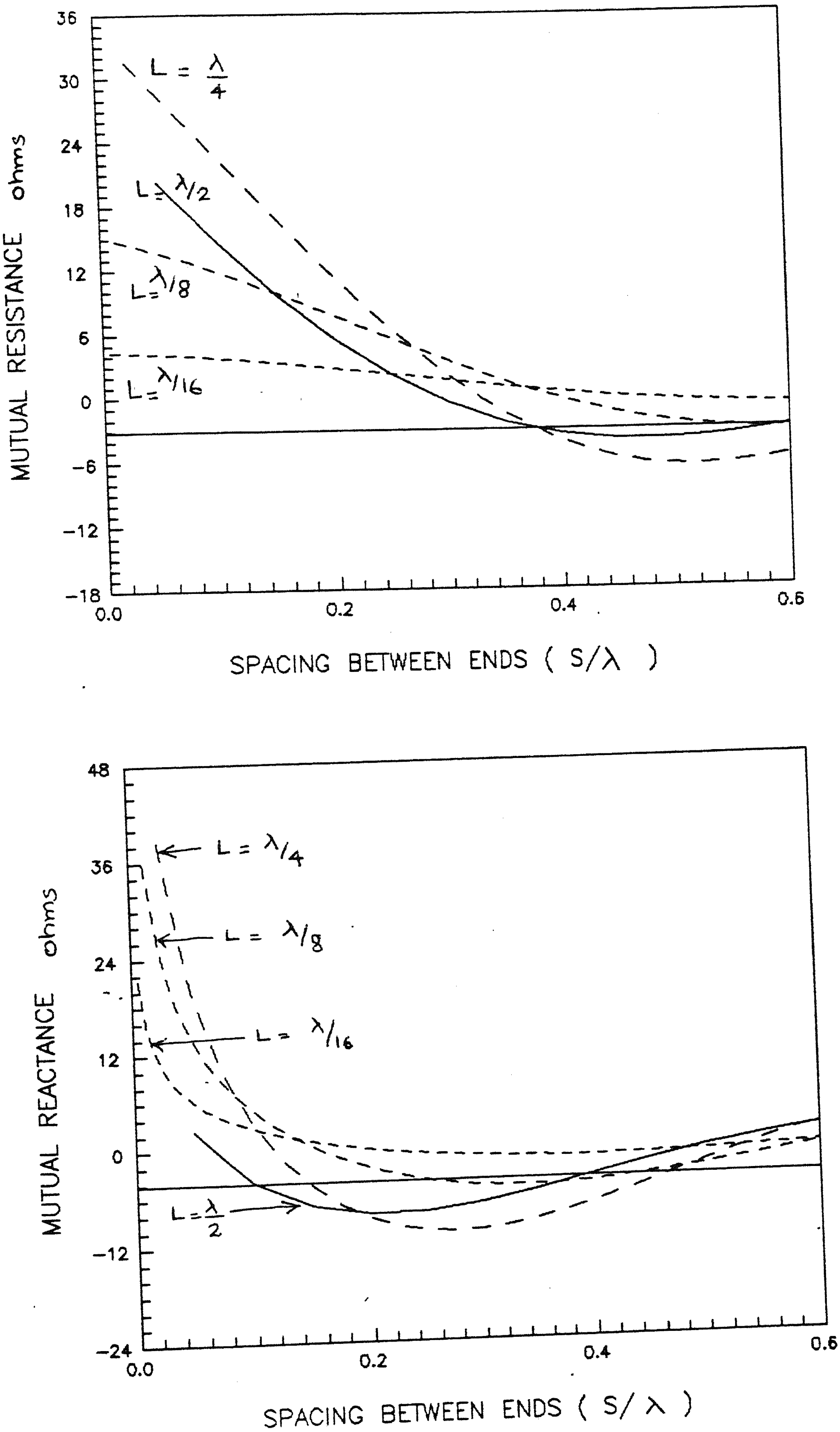


Fig 3.4 *Mutual coupling as a function of separation.*

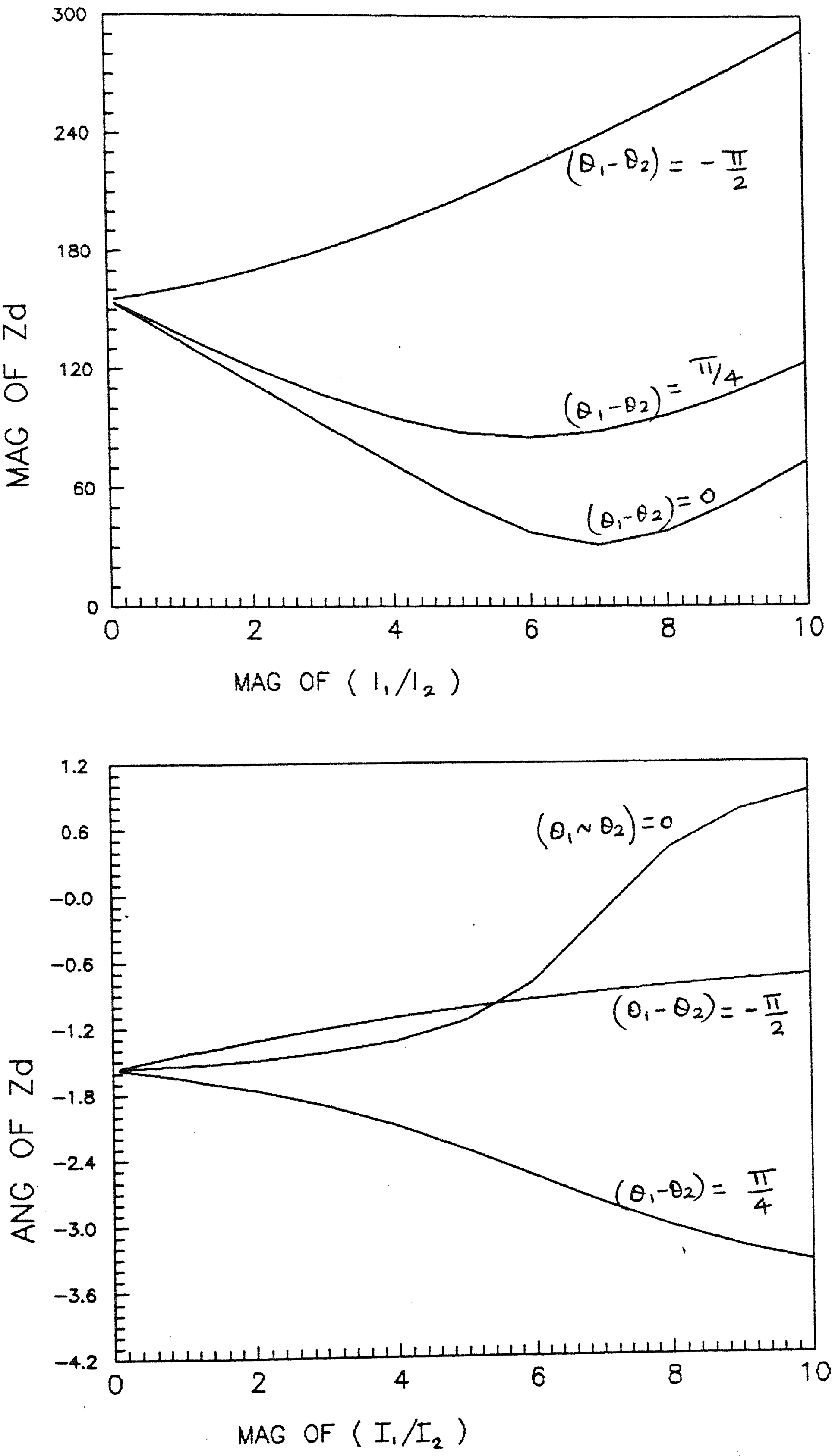
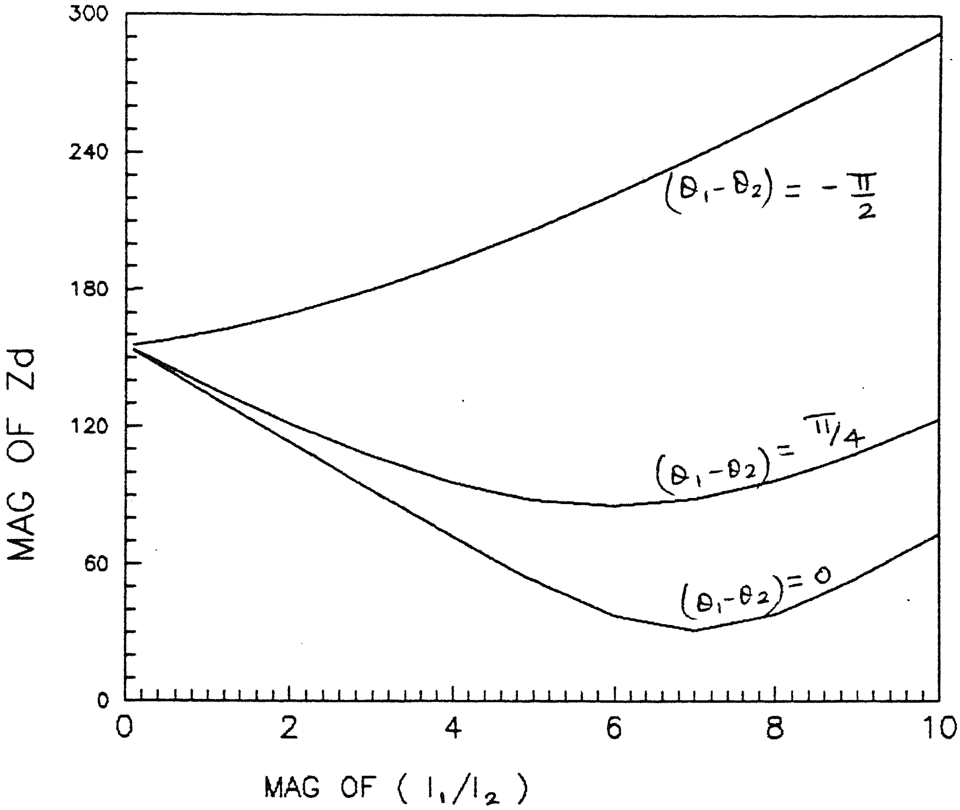


Fig 3.5 (a) *Driving point impedance as a function of input current at the terminals in two coupled antennas.*



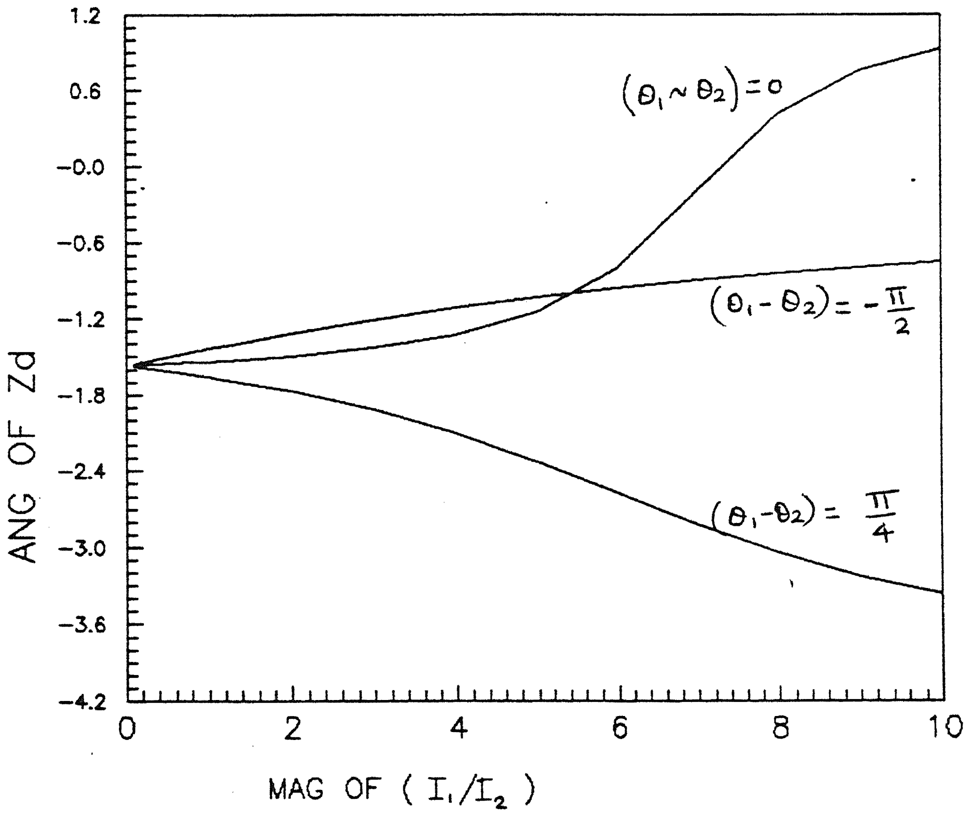
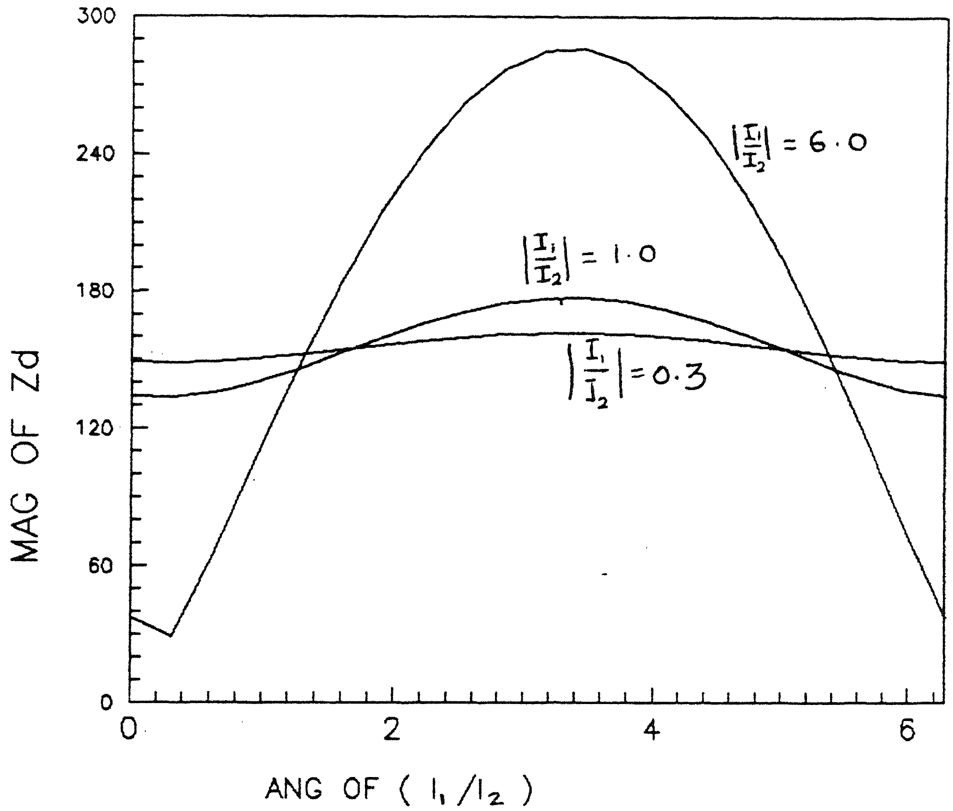


Fig 3.5 (a) *Driving point impedance as a function of input current at the terminals in two coupled antennas.*



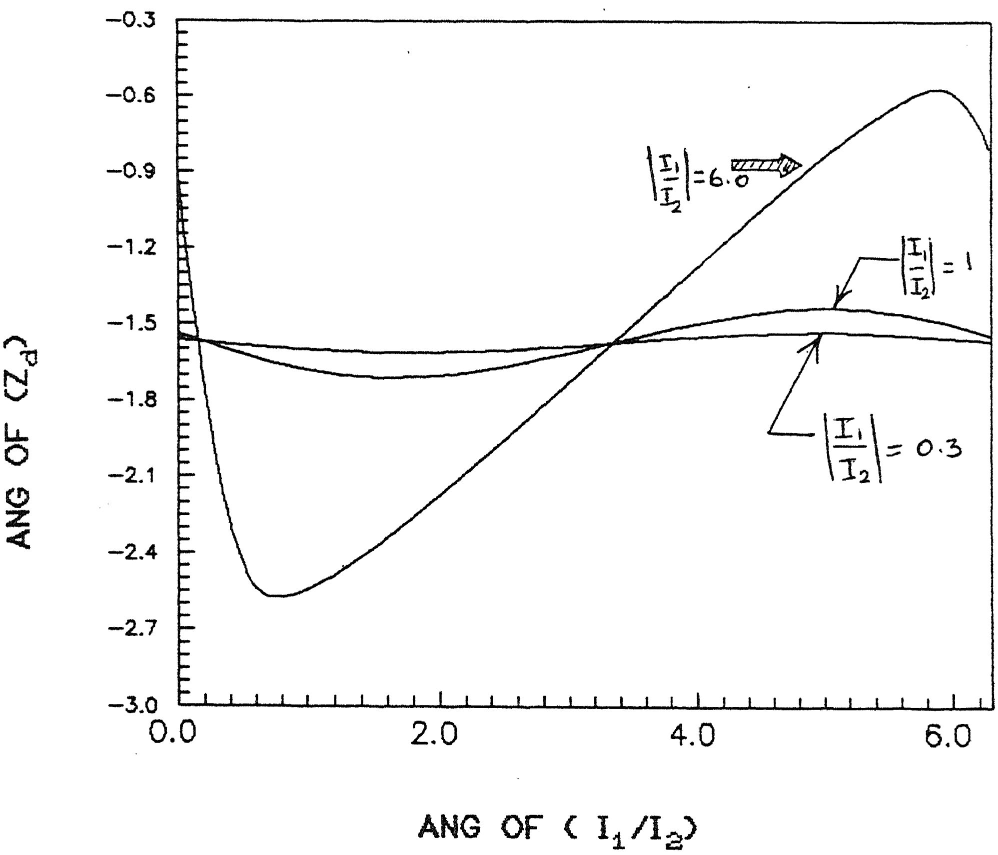


Fig 3.5 (b) *Driving point impedance as a function of input current at the terminals in two coupled antennas.*

$$Z_{id} = Z_{i1}\left[\frac{I_1}{I_i}\right] + Z_{i2}\left[\frac{I_2}{I_i}\right] + Z_{i3}\left[\frac{I_3}{I_i}\right] + \ldots + Z_{in}\left[\frac{I_n}{I_i}\right]$$

...(3.16)

3.6 Results and Discussion:

A program "Mutual.Pas" in Turbopascal version 4 implements equations (3.12) and (3.13) to calculate the coupling matrix or Z matrix of an n-element array of dipoles. For the sake of simplicity, important results for two coupled antennae are presented in fig.3.4(a) & 3.4(b). The coupling is significant for the closed spaced dipoles and diminishes rapidly with separation. That is, in an array of $\lambda/2$ dipoles only the immediate, adjacent antennae will have significant coupling. When the antenna dimensions and spacings are small compared to $\lambda/2$, coupling is stronger between several elements situated within a distance of $\lambda/2$.

Fig.3.5(a), fig.3.5(b) show the plot of variation of driving point impedance as a function of currents in the antenna. In the previous chapter, the individual element(electrically small

radiator) was found to be highly reactive and was difficult to match. A study of fig.3.5 reveal the role of the currents in the antenna(and implicitly the loads) to vary the driving point impedance suitable for matching. In the next chapter all these properties are used to synthesize the feed network

CHAPTER -4. ARRAY OF ELECTRICALLY SMALL RADIATORS

4.1 Introduction:

As mentioned earlier in chapter-1, the idea of selective feeding of a linear array of electrically small dipoles placed close to each other is utilized to control the current distribution on the antenna. Such a linear array sought to be designed to operate over a bandwidth (f_1 to f_2) with efficient radiation characteristics and constant beam width. Selectively exciting the array, needs a complicated feed network. This chapter discusses a method of arriving at the terminal parameters of such a feed network.

Fig 4.1 shows the arrangement of the linear array and the feed network. The box named antenna array encloses an array of n elements which can be looked at as a n-port network. These n-ports are fed by a (n+1)-port feed network. n-ports of feed network are connected to n terminals of antenna array and the (n+1)th port is connected to the source. It is at this port that we have to retain the match over the entire bandwidth, while obtaining the desired radiation characteristics.

A method is suggested here to synthesize the ~~f~~feed network terminal parameters for a given antenna array which is

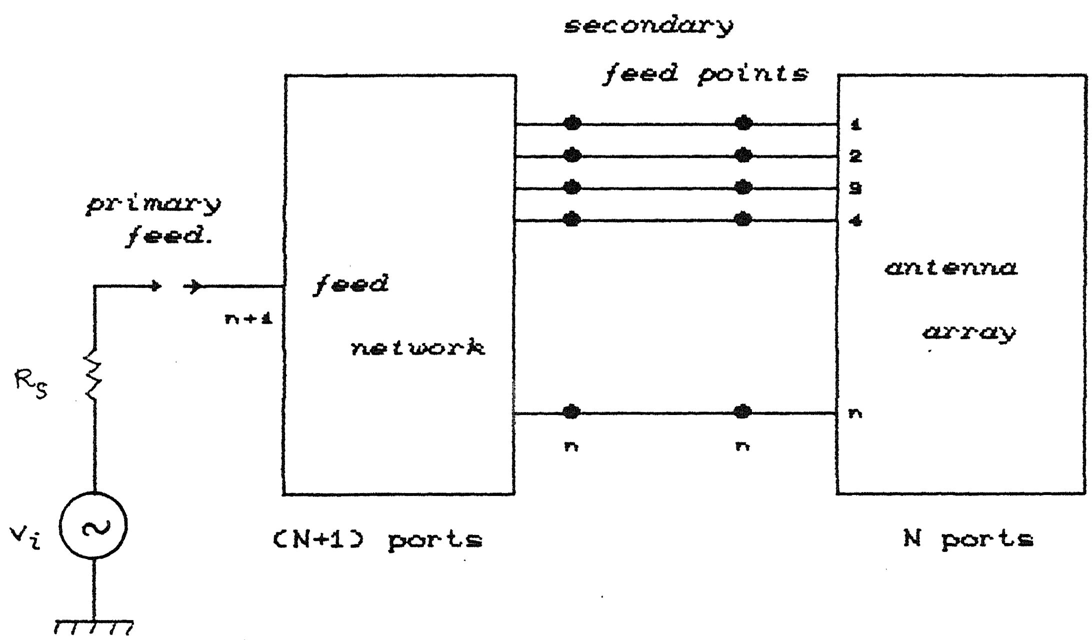


Fig 4.1. Closely spaced array of electrically small dipoles

and feed network.

characterized in terms of a coupling matrix ([S] or [Z]). The breakdown of the problem is as follows:

1. The Z-matrix characterizing the array of closely spaced electrically small radiators is determined using Induced EMF method as mentioned in section 3.3.

2. The terminal currents at each frequency are derived from the required current distribution, which gives the constant beam width pattern.

3. Given a set of currents at the n terminals of the antenna network, the driving point impedances are determined as explained in section 3.4.

4. From the terminal currents and the driving point impedances, the excitations necessary to sustain the given current distribution are calculated.

5. The requirements of the feed network is to provide the necessary secondary feed excitations (as calculated in step 4) and the driving point impedances (calculated in step 3) and to have minimum VSWR at the (n+1)th port; the primary feed point.

This chapter describes how to calculate the reflection coefficients at the (n+1)th port of feed network, and how to meet the requirements mentioned in step 5, while designing the feed network.

4.2 Reflection coefficient in a loaded network:

Fig 4.2 shows a_1, a_2 and b_1, b_2 as the incident and reflected waves at port 1 and port 2 of a two port network. a_1', a_2' and b_1', b_2' are the incident and reflected waves at load/source i.e., at port 1 and port 2. When the loads are connected to the networks we can equate

$$b_2 = a_2' \qquad\qquad a_2 = b_2'$$

$$b_1 = a_1' \qquad\qquad a_1 = b_1'$$

Therefore the load reflection coefficient can be written as:

$$\Gamma_L = b_2' / a_2' = a_2 / b_2 \qquad \ldots(4.1)$$

This network can be characterized in terms of a 2x2 S-parameter matrix. When there is no coupling (or no interaction between the ports) the S matrix of the feed network is diagonal. Therefore, the reflection coefficient at each port is directly the self reflection at that port i.e., S_{11} at 11' and S_{22} at 22'.

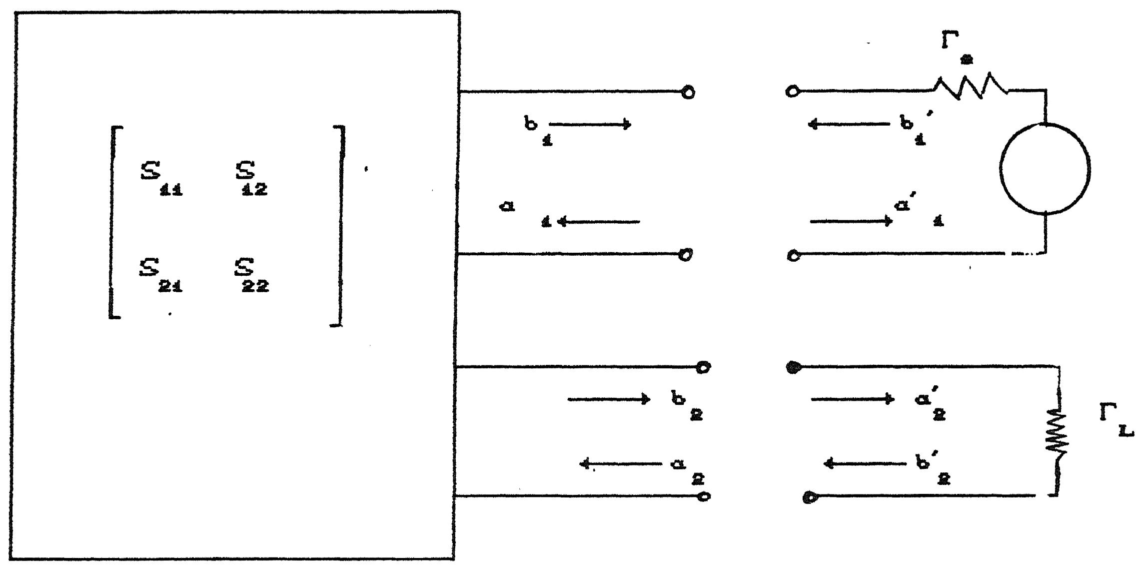


Fig 4.2 A Loded Two Port Network.

When $S_{12} \neq 0$ and $S_{21} \neq 0$, the reflection coefficient at 11' and 22' are quite different from S_{11} and S_{22} and are given by equations 4.2 and 4.3.

$$\Gamma_1 = S_{11} + S_{12}(S_{21}\Gamma_L/(1-S_{22}\Gamma_L)) \qquad \ldots(4.2)$$

$$\Gamma_2 = S_{22} + S_{21}(S_{12}\Gamma_S/(1-S_{11}\Gamma_S)) \qquad \ldots(4.3)$$

In a two-port network, the calculation of input reflection coefficient at port 1, when port 2 is terminated by a load of reflection coefficient Γ_L, is fairly simple. Computer evaluation is needed for a multi port network. In the present problem the load terminations are not simple impedances, but is an antenna array, which is characterized by a n-port network scatter matrix. The reflection coefficient at each port of the network is arrived at by solving a set of equations iteratively.

Consider a 3-port network feeding a two port load as shown in fig 4.3. ${}^{i}\Gamma_{j}$ refers to the reflection coefficient at port j of network i. the primed S-parameters correspond to the second network and the unprimed S-parameters correspond to the first network. The reflection coefficient at port 3 of network 1 can be expressed as,

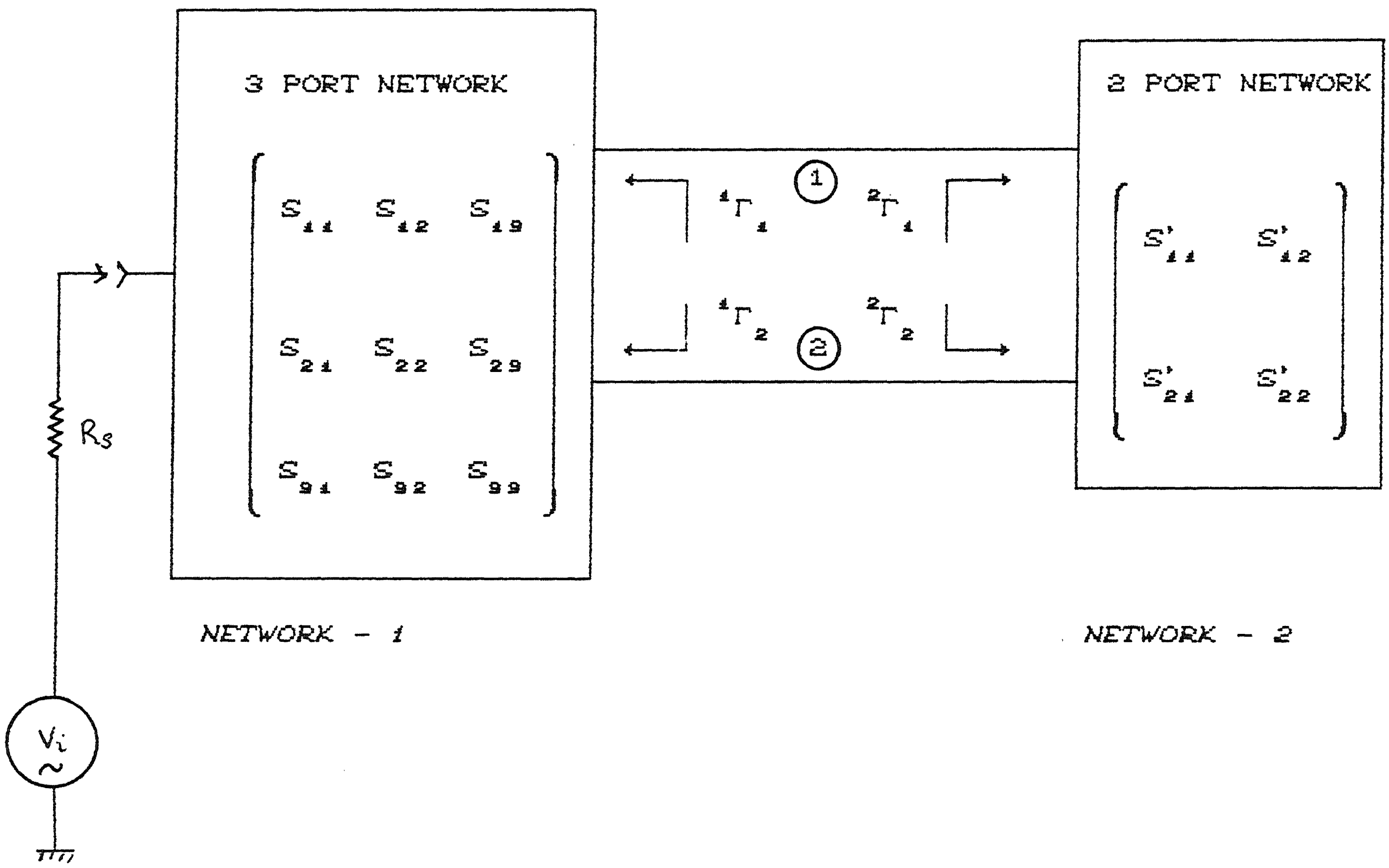


Fig 4.3. *3-port network loaded by a two port load.*

$$ {}^{1}\Gamma_{3} = S_{33} + S_{31}(a_1/a_3) + S_{32}(a_2/a_3) \qquad \ldots(4.4)$$

where,

$$\left[\frac{a_2}{a_3}\right] = \frac{{}^{2}\Gamma_{2}\left[S_{23} + \dfrac{S_{13}\,S_{21}\,{}^{2}\Gamma_{1}}{1 - S_{11}\,{}^{2}\Gamma_{1}}\right]}{1 - {}^{2}\Gamma_{2}\left[S_{22} + \dfrac{S_{12}\,S_{21}\,{}^{2}\Gamma_{1}}{1 - S_{11}\,{}^{2}\Gamma_{1}}\right]} \qquad \ldots(4.5)$$

and

$$\left[\frac{a_1}{a_3}\right] = \frac{{}^{2}\Gamma_{1}}{1 - S_{11}\,{}^{2}\Gamma_{1}}\left[S_{13} + S_{12}\left[\frac{a_2}{a_3}\right]\right] \qquad \ldots(4.6)$$

The ratios a_2/a_3, a_1/a_3 are derived in appendix-I.

To evaluate the equation (4.4), the reflection coefficient of $({}^{2}\Gamma_{1}, {}^{2}\Gamma_{2})$ network 2 should be known. If the ports 2 and 3 of network 1 are loaded by mutually independent loads, meaning S-parameter of network 2 is diagonal,$({}^{2}\Gamma_{1}, {}^{2}\Gamma_{2})$ are independent of each other and equation (4.4) cán be evaluated directly.

But, if the network 2 is a coupled load, (say two antenna elements in an array) ${}^{2}\Gamma_{1}$, ${}^{2}\Gamma_{2}$ are also unknowns. The reflection coefficient ${}^{2}\Gamma_{1}$, ${}^{2}\Gamma_{2}$ are functions of S-parameters of network 2 and reflection coefficient of network 1, ${}^{1}\Gamma_{1}$, ${}^{1}\Gamma_{2}$, ${}^{1}\Gamma_{3}$. The expressions for ${}^{1}\Gamma_{1}$, ${}^{1}\Gamma_{2}$ are similar to equation (4.4) to equation (4.6). For network-2 the reflection coefficients are,

$${}^{2}\Gamma_{1} = (b'_{1}/a'_{1}) = S'_{11} + S'_{12}(a'_{2}/a'_{1}) \qquad \text{...(4.7)}$$

$${}^{2}\Gamma_{2} = (b'_{2}/a'_{2}) = S'_{22} + S'_{21}(a'_{1}/a'_{2}) \qquad \text{...(4.8)}$$

where,

$$\left[\frac{a'_{1}}{a'_{2}}\right] = \frac{{}^{1}\Gamma_{1}\, S'_{12}}{1 - S'_{11}\,{}^{1}\Gamma_{1}}$$

$$\left[\frac{a'_{2}}{a'_{1}}\right] = \frac{{}^{1}\Gamma_{2}\, S'_{21}}{1 - S'_{22}\,{}^{1}\Gamma_{2}}$$

The reflection coefficient of network 1 ${}^{1}\Gamma_{1}$, ${}^{1}\Gamma_{2}$

$${}^{1}\Gamma_{1} = S_{11} + S_{12}(a_{2}/a_{1}) + S_{13}(a_{3}/a_{1}) \qquad \text{...(4.9)}$$

$${}^{1}\Gamma_{2} = S_{22} + S_{21}(a_{1}/a_{2}) + S_{23}(a_{3}/a_{2}) \qquad \text{...(4.10)}$$

$${}^{1}\Gamma_{3} = S_{33} + S_{31}(a_{1}/a_{3}) + S_{32}(a_{2}/a_{3}) \qquad \text{...(4.11)}$$

The incident wave ratios (a_i/a_j) of network 1 are functions of S-parameters of network 1 and reflection coefficients of network 2 (Also vice versa for (a_i/a_j) of network 2). Hence the equations to be solved are two sets of coupled simultaneous equations. The ratios (a_2/a_1), (a_3/a_1) etc.. for a n-port network is lengthy, but can be calculated using a recursive relationship. To obtain the recursive relationship we examine the ratios (a_2/a_1) for a two port, three port and four port networks. These are given by the expressions derived in Appendix-I.

Two port network :

$$\left[\frac{a_2}{a_1}\right] = \frac{\Gamma_2 S_{21}}{1 - S_{22}\Gamma_2} \qquad \text{...(4.12)}$$

Three port network :

$$\left[\frac{a_2}{a_1}\right] = \frac{\Gamma_2\left[S_{21} + \dfrac{S_{31} S_{23} \Gamma_3}{1 - S_{33}\Gamma_3}\right]}{1 - \Gamma_2\left[S22 + \dfrac{S_{32} S_{23} \Gamma_3}{1 - S_{33}\Gamma_3}\right]} \qquad \text{...(4.13)}$$

<u>Four port network</u> :

$$\left[\frac{a_2}{a_1}\right] = \frac{\Gamma_2\left[S_{21} + \dfrac{S_{\underset{4}{\cancel{3}}1}\;S_{2\underset{4}{\cancel{3}}}\;\Gamma_{\underset{4}{\cancel{3}}}}{1 - S_{\underset{44}{\cancel{33}}}\;\Gamma_{\underset{4}{\cancel{3}}}} + A\right]}{1 - \Gamma_2\left[S22 + \dfrac{S_{\overset{4}{\cancel{3}}2}\;S_{2\overset{4}{\cancel{3}}}\;\Gamma_{\cancel{34}\,4}}{1 - S_{\underset{44}{\cancel{33}}}\;\Gamma_{\underset{4}{\cancel{3}}}} + B\right]} \qquad \ldots(4.14)$$

Where A and B are given by,

$$A = \frac{\Gamma_3\left[S_{23} + \dfrac{S_{43}\;S_{24}\;\Gamma_4}{1 - S_{44}\;\Gamma_4}\right]\left[S_{31} + \dfrac{S_{34}S_{41}\;\Gamma_4}{1 - S_{44}\;\Gamma_4}\right]}{1 - \Gamma_3\left[S_{33} + \dfrac{S_{43}\;S_{34}\;\Gamma_4}{1 - \Gamma_4\;S_{44}}\right]} \qquad \ldots(4.15)$$

$$B = \frac{\Gamma_3\left[S_{23} + \dfrac{S_{43}\;S_{24}\;\Gamma_4}{1 - S_{44}\;\Gamma_4}\right]\left[S_{32} + \dfrac{S_{34}S_{42}\;\Gamma_4}{1 - S_{44}\;\Gamma_4}\right]}{1 - \Gamma_3\left[S_{33} + \dfrac{S_{43}\;S_{34}\;\Gamma_4}{1 - \Gamma_4\;S_{44}}\right]} \qquad \ldots(4.16)$$

In general, for an n-port network, the recursive relationship for the ratio (a_m/a_1) can be written as,

$$ {}^{n}\left[\frac{a_m}{a_1}\right] = \frac{\Gamma_m}{1 - \Gamma_m \left(S_{m,m} \right)_m} \left[\sum_{q=1}^{m-1} \left(S_{m,q} \right)_m \left(\frac{a_q}{a_1} \right) \right] \quad ; \; 1 < m \leq n \qquad \ldots (4.17) $$

where n represents the number of ports.

$(S_{i,j})_x$ is given by,

$$ \left(S_{i,j} \right)_x = \left[\left(S_{i,j} \right)_{x+1} + \frac{\left(S_{i,x+1} \right)_{x+1} \left(S_{x+1,i} \right)_{x+1}}{1 - \Gamma_{x+1} \left(S_{x+1,x+1} \right)_{x+1}} \right] \quad ; \; x < \cancel{m}\, n \qquad \ldots (4.18) $$

$$ (S_{i,j})_x = S_{i,j} \quad ; \quad x = n \qquad \ldots (4.19) $$

4.3. Method of Synthesis of Feed Network Characteristics:

The configuration of the antenna system and the feed network are shown in fig 4.1. In the introduction to this chapter, we have discussed some preliminary points about feed network. Section 4.2 analyzed an n-port network, by which one can compute the reflection coefficient at any port, if the S-parameters characterizing the network and the load reflection coefficient are known.

The requirement of the feed network is to provide proper excitations at each frequency to sustain the desired current distribution on the antenna. To ensure this, the driving point impedances and excitations have to be equal to that calculated earlier. In addition to this we have to ensure that the input reflection coefficient is minimum at the primary feed point. Thus, the problem is one of minimizing the input reflection coefficient at $(n+1)^{th}$ port subject to the constraints of providing the desired driving point impedances and excitations to achieve the assumed current distribution on the antenna.

The analysis, given so far, calculates the input reflection coefficient, driving point impedances and excitations, given the S-parameters of the antenna and the feed network. The synthesis

problem is an inverse problem where we have to arrive at the S-matrix of the feed network to give the proper driving point impedances and minimum reflection coefficient at the primary feed point. This can only be solved iteratively, using minimization algorithms. In this work, a pattern search method is used to arrive at the S-matrix of the feed network.

In this method, an initial S-matrix is assumed for the feed network and its elements are successively updated based on some error criteria, till the error is minimum possible. the strategy of pattern search[10][11] states that progress towards a minimum is accomplished by learning the behavior of a function in and around a defined base point, and based on this information, a move is made. This search technique involves two type of moves viz., Explore move and Pattern move. In the exploring move each variable of optimization is perturbed and the error is evaluated in the vicinity of base point. Then a vector in error space is generated which terminates on a point of lower error. If no lower error is noted, a smaller step is used (points closer to base point are searched).

The other move is pattern move. Pattern move establishes the new point found by the explore move and attempts to take a pattern step. The definition of a pattern move can be best

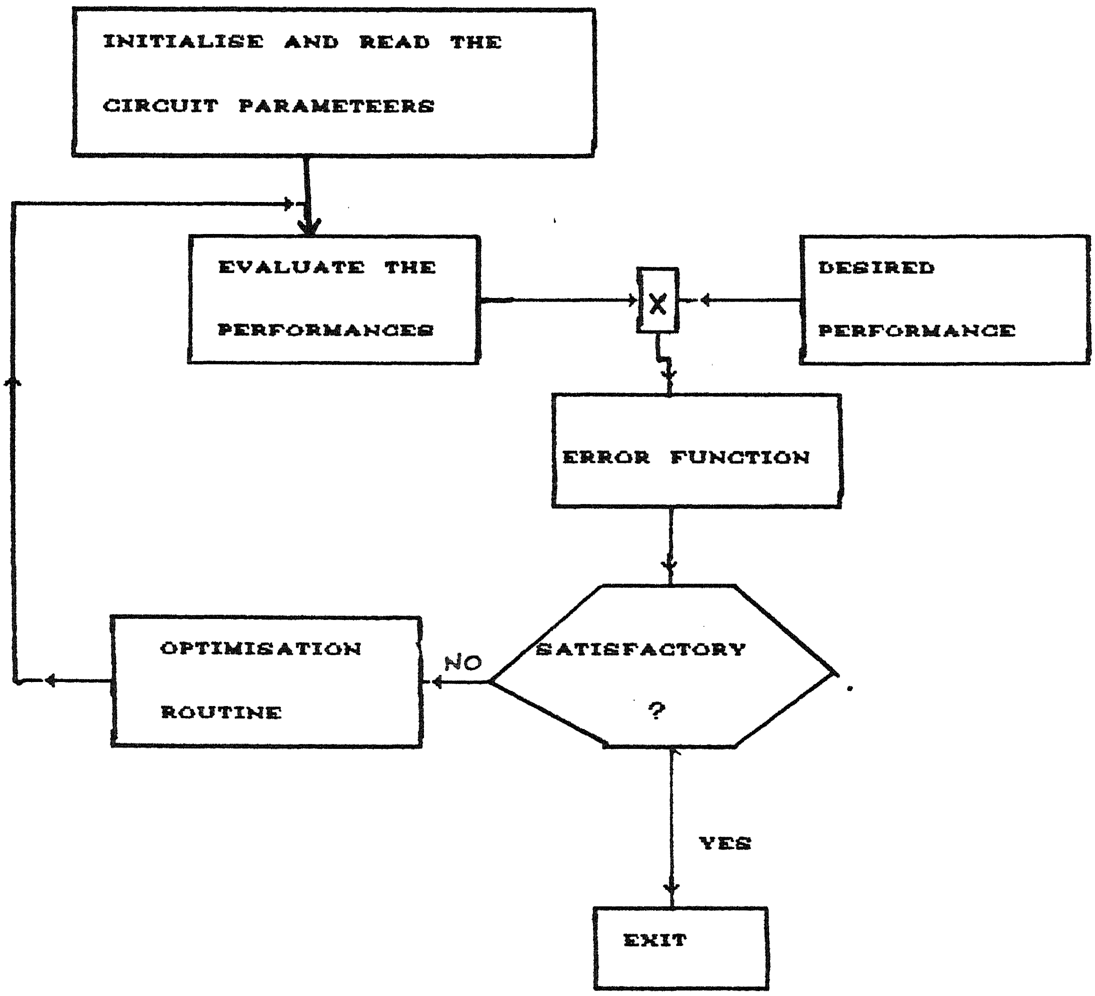


Fig 4.4 Structure of an *Optimal Seeking CAD* [OSCAD]

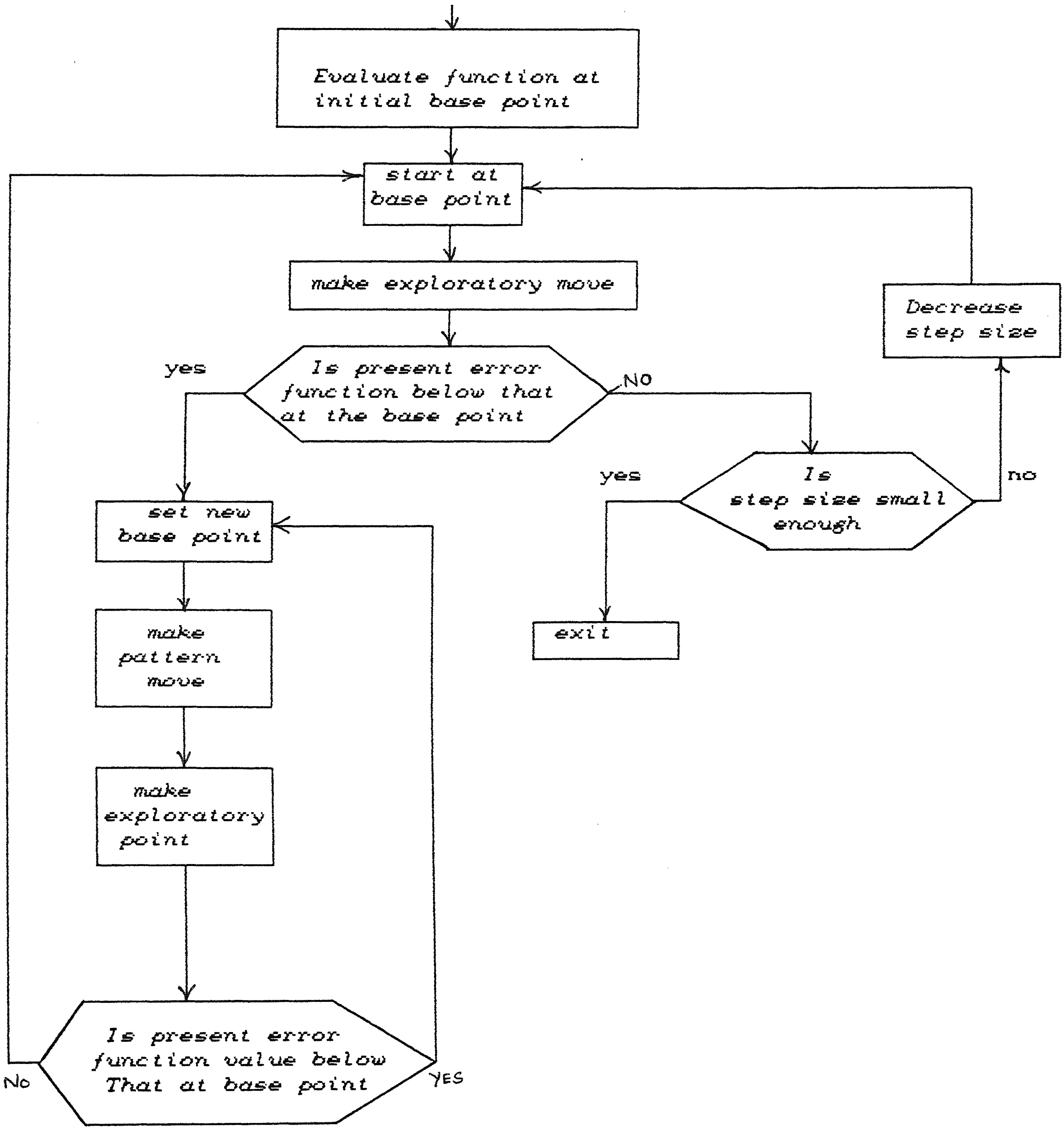


Fig 4.5 Flow chart of search technique implimented in DEMON.

described by saying that it is an intelligent move based on some recent observations (exploratory move) of the behavior of the function. In a pattern move a large step in the direction of the vector, generated by explore move, is taken. If this is successful in optimizing the function, a larger step is taken and so on, until a failure occurs. Fig(4.4) and Fig(4.5) show the flow chart of the optimization algorithm.

4.4 Synthesis of Feed Network Scatter Matrix:

The formulation of [Z] matrix of the antenna array is already treated in section 3.4. Here we assume that [Z] matrix of the antenna array, driving point impedances [Z_d] as well as the excitation voltages [V], are known. As a first step, the [Z] matrix is converted to a corresponding [S] matrix by the transformation equations. This is a strait forward procedure and is briefly explained here .

$$[E] = [Z]\ [I] \qquad \ldots(4.20)$$

where E is the voltage vector~~s~~, I is the current vector. Using the incident wave 'a' and reflected wave 'b' terminology E and I can be expressed as (a+b) and (a-b). Therefore,

$$(a+b) = Z\,(a-b) \qquad \ldots(4.21)$$

if U is the Unitary matrix then,

$$S = b/a = [U+Z]^{-1}\,[Z-U] \qquad \ldots(4.22)$$

The driving point impedance at the antenna terminals are converted to corresponding reflection coefficient using

$$\Gamma_i = \frac{Z_{di} - Z_o}{Z_{di} + Z_o} \qquad \ldots(4.23)$$

Now, consider the calculation of the reflection coefficient at various ports of an n-port network, when it is loaded by another n-port network. Fig 4.6 shows the configuration used for calculating reflection coefficient of m^{th} port of the n-port network. This involves solution of two sets of coupled simultaneous equations, which are given below. The subscript 1,2 on S-parameters denote the network 1 or network 2.

For network 1, the reflection coefficients are,

$${}^{1}\Gamma_1 = {}^{1}S_{11} + {}^{1}S_{12}{}^{1}\left(\frac{a_2}{a_1}\right) + {}^{1}S_{13}{}^{1}\left(\frac{a_3}{a_1}\right) + {}^{1}S_{14}{}^{1}\left(\frac{a_4}{a_1}\right) + \ldots + {}^{1}S_{1n}{}^{1}\left(\frac{a_n}{a_1}\right) \qquad \ldots(4.24)$$

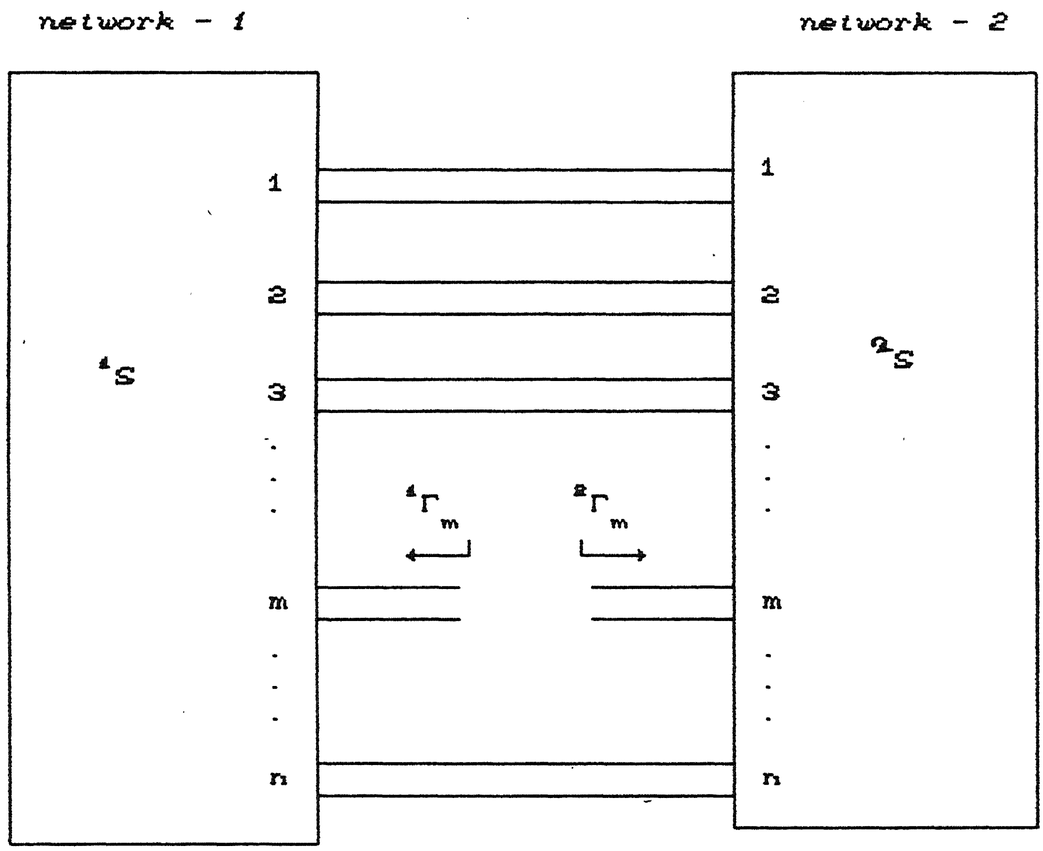


Fig 4.6 Reflection coefficient of m^{th} port of an n-port network

$$ {}^{1}\Gamma_{2} = {}^{1}S_{21}\,{}^{1}\!\left[\frac{a_1}{a_2}\right] + {}^{1}S_{22} + {}^{1}S_{23}\,{}^{1}\!\left[\frac{a_3}{a_2}\right] + {}^{1}S_{24}\,{}^{1}\!\left[\frac{a_4}{a_2}\right] + \ldots + {}^{1}S_{2n}\,{}^{1}\!\left[\frac{a_n}{a_2}\right] \qquad \ldots(4.25) $$

$$ {}^{1}\Gamma_{3} = {}^{1}S_{31}\,{}^{1}\!\left[\frac{a_1}{a_3}\right] + {}^{1}S_{32}\,{}^{1}\!\left[\frac{a_2}{a_3}\right] + {}^{1}S_{33} + {}^{1}S_{34}\,{}^{1}\!\left[\frac{a_4}{a_3}\right] + \ldots + {}^{1}S_{3n}\,{}^{1}\!\left[\frac{a_n}{a_3}\right] \qquad \ldots(4.26) $$

$$ {}^{1}\Gamma_{n} = {}^{1}S_{n1}\,{}^{1}\!\left[\frac{a_1}{a_n}\right] + {}^{1}S_{n2}\,{}^{1}\!\left[\frac{a_2}{a_n}\right] + {}^{1}S_{n3}\,{}^{1}\!\left[\frac{a_3}{a_n}\right] + {}^{1}S_{n4}\,{}^{1}\!\left[\frac{a_4}{a_n}\right] + \ldots + {}^{1}S_{nn} \qquad \ldots(4.27) $$

For network 2 the reflection coefficients are,

$$ {}^{2}\Gamma_{1} = {}^{2}S_{11} + {}^{2}S_{12}\,{}^{2}\!\left[\frac{a_2}{a_1}\right] + {}^{2}S_{13}\,{}^{2}\!\left[\frac{a_3}{a_1}\right] + {}^{2}S_{14}\,{}^{2}\!\left[\frac{a_4}{a_1}\right] + \ldots + {}^{2}S_{1n}\,{}^{2}\!\left[\frac{a_n}{a_1}\right] \qquad \ldots(4.28) $$

$$ {}^{2}\Gamma_{2} = {}^{2}S_{21}\,{}^{2}\!\left[\frac{a_1}{a_2}\right] + {}^{2}S_{22} + {}^{2}S_{23}\,{}^{2}\!\left[\frac{a_3}{a_2}\right] + {}^{2}S_{24}\,{}^{2}\!\left[\frac{a_4}{a_2}\right] + \ldots + {}^{2}S_{2n}\,{}^{2}\!\left[\frac{a_n}{a_2}\right] \qquad \ldots(4.29) $$

$$ {}^{2}\Gamma_{3} = {}^{2}S_{31}^{\,2}\left(\frac{a_1}{a_3}\right) + {}^{2}S_{32}^{\,2}\left(\frac{a_2}{a_3}\right) + {}^{2}S_{33} + {}^{2}S_{34}^{\,2}\left(\frac{a_4}{a_3}\right) + \ldots + {}^{2}S_{3n}^{\,2}\left(\frac{a_n}{a_3}\right) \qquad \ldots(4.30) $$

$$ {}^{2}\Gamma_{n} = {}^{2}S_{n1}^{\,2}\left(\frac{a_1}{a_n}\right) + {}^{2}S_{n2}^{\,2}\left(\frac{a_2}{a_n}\right) + {}^{2}S_{n3}^{\,2}\left(\frac{a_3}{a_n}\right) + {}^{2}S_{n4}^{\,2}\left(\frac{a_4}{a_n}\right) + \ldots + {}^{2}S_{nn} \qquad \ldots(4.31) $$

The ratios (a_i/a_j) can be calculated using the expressions derived earlier in section 4.2, equation (4.17) to equation (4.19).

The expressions (a_i/a_j) of network 1 involves the reflection coefficients of network 2 (${}^{2}\Gamma_1$, ${}^{2}\Gamma_2, \ldots, {}^{2}\Gamma_n$), which are given by the rations (a_i/a_j) of network 2. As the ratio (a_i/a_j) also involves the reflection coefficient of network 1 these equations may be solved iteratively to obtain the reflection coefficient of every port of each network. The iterative process can be briefly described as follows.

The reflection coefficient at port 'm' is a sum of the self reflection coefficients S_{mm} at port m and the scatter parameters S_{mi} between the m^{th} port and all the other ports

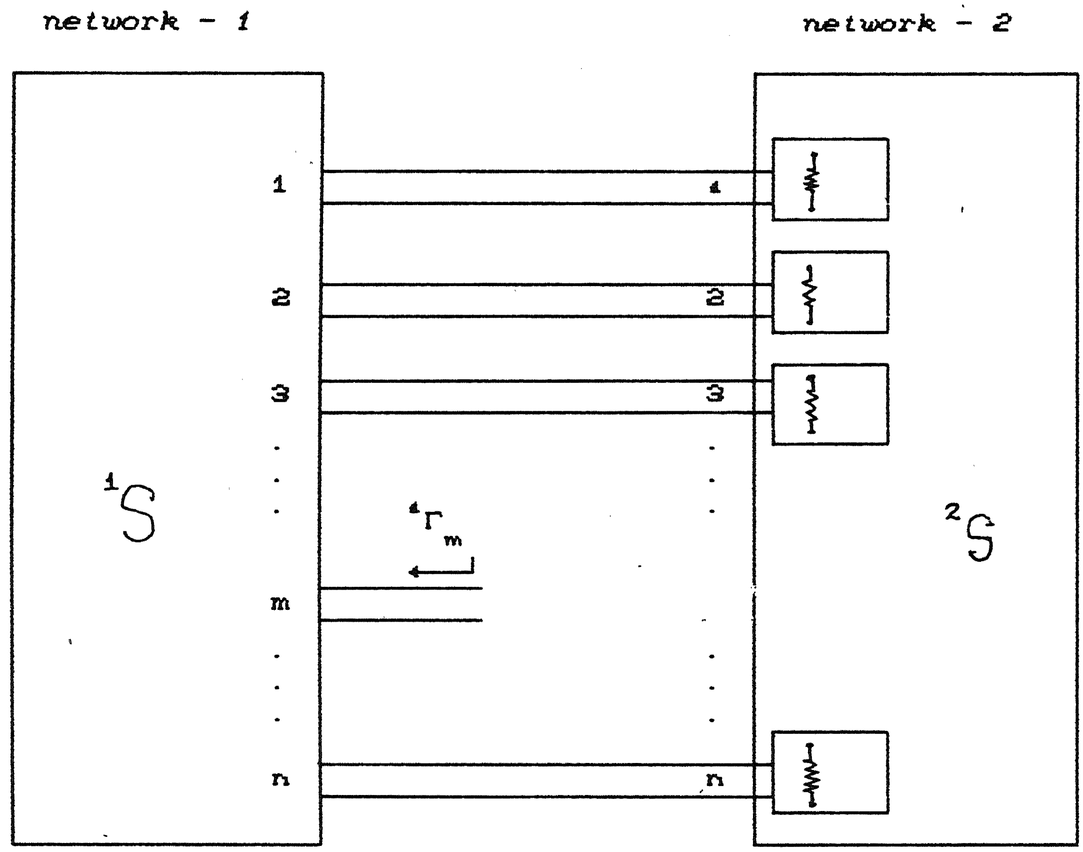


Fig 4.7 Configuration for calculating reflection at port m.

weighted by ratios (a_i/a_m). Hence a first order approximation can be made that the reflection coefficient is only due to the self reflection ie., S_{mm} (fig 4.7). This approximation serves as the starting point for the iterative solution. The algorithm is,

(1) Assume the load reflection coefficient of network 1 as S_{ii} of network 2. (This is ${}^{1}\Gamma_i$, when all non diagonal elements of network 2 are zero).

(2) Solve for the reflection coefficient of all the ports of network 1 with the terminating load reflections (${}^{2}\Gamma_i$) as calculated in step (1) in first iteration and later from step (3).

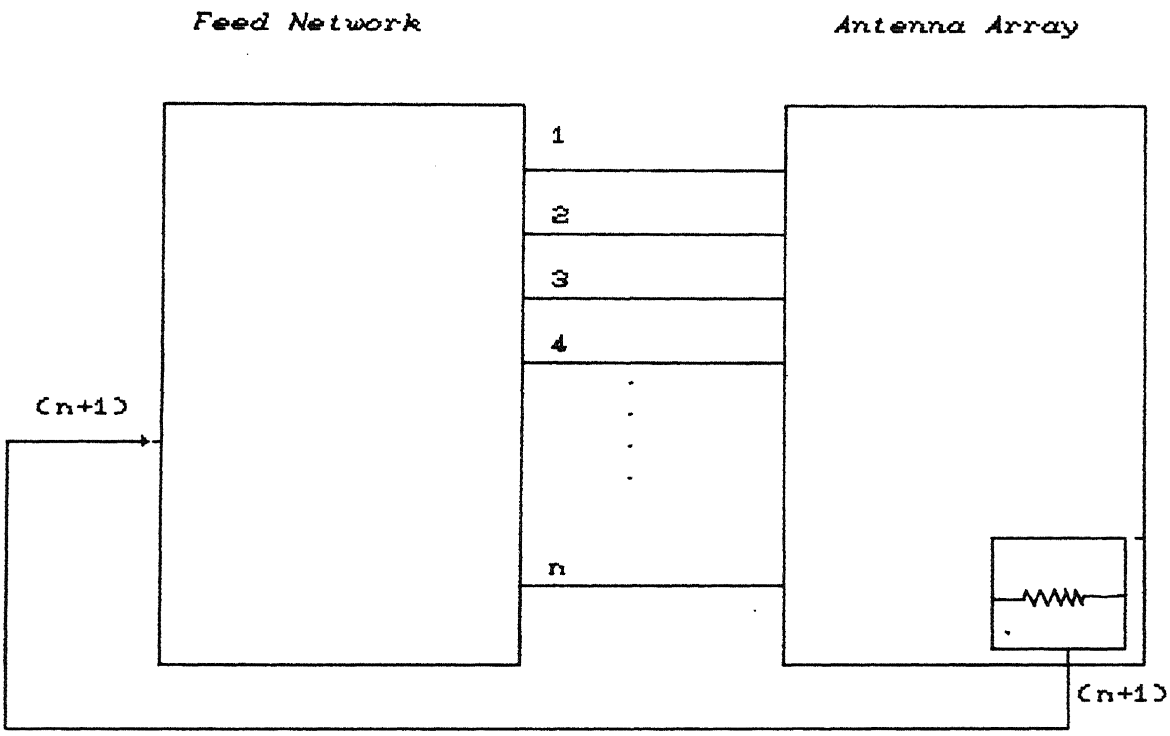
(3) Now, take the calculated reflection coefficient of network 1 from step (2) as the terminating load reflections connected to network 2 and solve for the reflection coefficients of network 2.

(4) Iterate between steps (2) and (3), till the difference between two successive solutions of reflection coefficient of both the networks are within a specified error limit.

So far it has been discussed how to find out the reflection coefficient of an n-port network connected to another n-port network. But in our case, the antenna is an array of n

elements characterised by n-port network (n x n matrix), whereas the feed network has to be an (n+1) port network. (n-ports connected to the antenna ports and $(n+1)^{th}$ port connected to the source). In order to treat both the networks as a general P-port network, where P is the number of elements in the antenna array plus one, the antenna matrix can be augmented by adding an additional row and column with all zeros. This is equivalent to an $(n+1)^{th}$ port, which is uncoupled to any other ports in the network, and offers a matched termination. The primary feed point of feed network is connected to this $(n+1)^{th}$ port of antenna network (fig 4.8).

The voltage column matrix calculated in chapter-3 should be next transferred in a suitable form so that it can be represented by the feed network S-parameters. The square of transfer S-parameters from primary feed point, $(n+1)^{th}$ port, to secondary feed points (1 to n) ie., $S^2_{i,(n+1)}$, of the feed network represents the power transferred from source port to antenna elements. From the property of Scattering parameters in a linear passive non dissipative network, the power input at a port is equal to the sum of power transferred to all the other ports. Hence summing up the normalised power at secondary feed points and equating to one, the normalised power at primary feed point, the driving point voltages can be scaled down as given below:



$$\begin{bmatrix} 11 & 12 & 13 & \cdots & 1n & 0 \\ 21 & 22 & 23 & \cdots & 2n & 0 \\ 31 & 32 & 33 & \cdots & 3n & 0 \\ \vdots & & & & & \\ n1 & n2 & n3 & \cdots & nn & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Fig 4.8 Augmented antenna network & S-matrix.

$$S^2_{I,(N+1)} = \frac{V_I^2}{\sum_{J=1}^{N} V_J^2} \qquad \ldots(4.32)$$

Now, the augmented (n+1 x n+1) S-matrix of antenna is known and the $(n+1)^{th}$ row and column of feed network is known. The rest of the S-parameters of feed network are given a starting value and the reflection coefficient at various ports [Γ_i to Γ_{n+1}] are calculated as explained earlier.

Our object is to make these reflection coefficients looking into antenna ports [Γ_1 to Γ_n] equal to the corresponding reflection coefficient calculated earlier, which will sustain the necessary current distribution . To obtain a match at the primary feed point, the reflection coefficient at $(n+1)^{th}$ port should be zero. So, we can form the objective function as follows.

$$\Phi = W_1 \quad \Gamma_{N+1}^{\quad 2} + \sum_{I=1}^{N} \Gamma_{DI} - \Gamma_I^{\quad 2} \qquad \ldots(4.33)$$

where W_1 is a weighting term used to adjust the relative emphasis between a match to the source and a match to the driving point impedances. The objective function ϕ will be zero when the

feed network S-parameters meet our requirements. The objective function ϕ is minimised using the pattern search algorithm.

The location of antenna elements and symmetry in the configuration are utilised to reduce the number of variables of optimization. For example, an array of seven elements as in Fig 4.9, will have identical environment for elements one and seven, two and six, three and five. The feed network is a reciprocal passive network. So, $S_{ij} = S_{ji}$. From the required excitations at antenna terminals, $S_{i,n+1}$ parameters are known. Now, due to the identical array environment of different elements and a symmetrical current distribution about the center element, $S_{ij} = S_{n-j,n-i}$. Considering all these into account, the total number of variables of optimisation are $[(n+1)/2]^2$.

4.5 Results and Conclusion:

This chapter started with the requirements of the feed network to excite an array of electrically small radiators. The problem is broken down to analysing an n-port network, under loaded conditions and then using the reverse process to synthesise the feed network terminal parameters. the method suggested in this chapter is implemented in an optimisation routine. As an example, an array of five elements is designed and the results are presented.

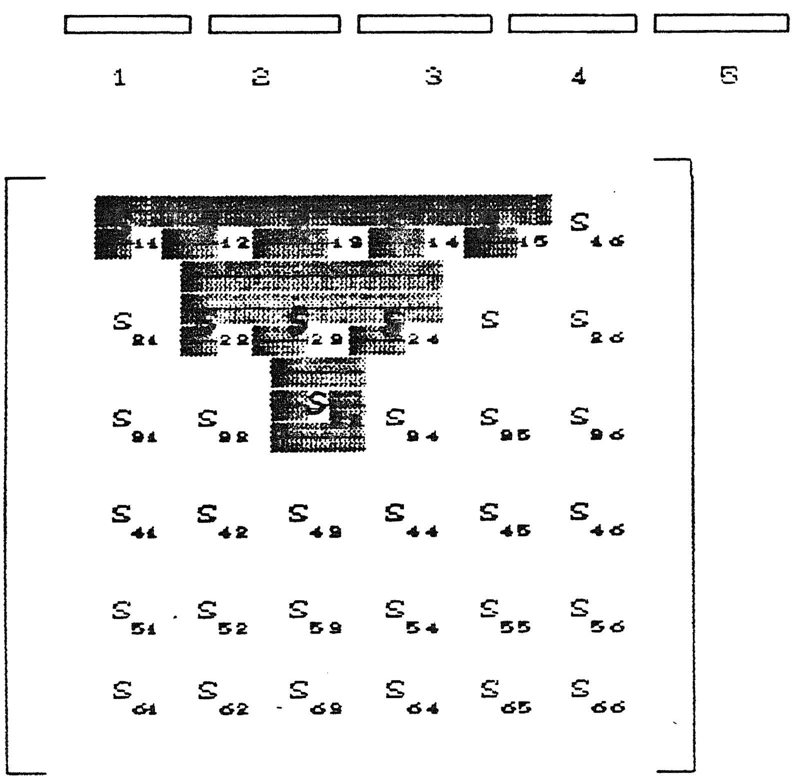


Fig 4.8 Array of $\overset{5}{\nabla}$ elements.

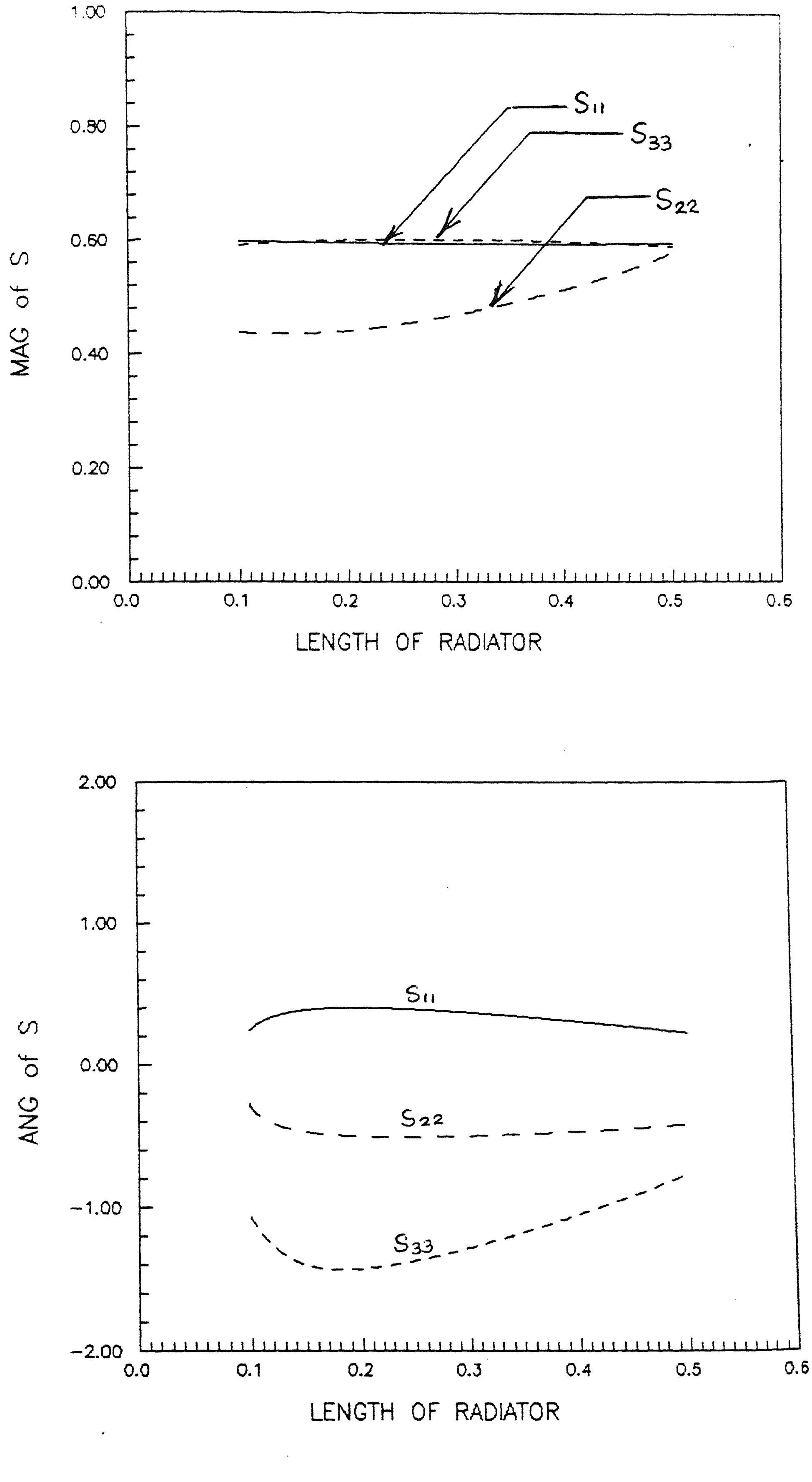
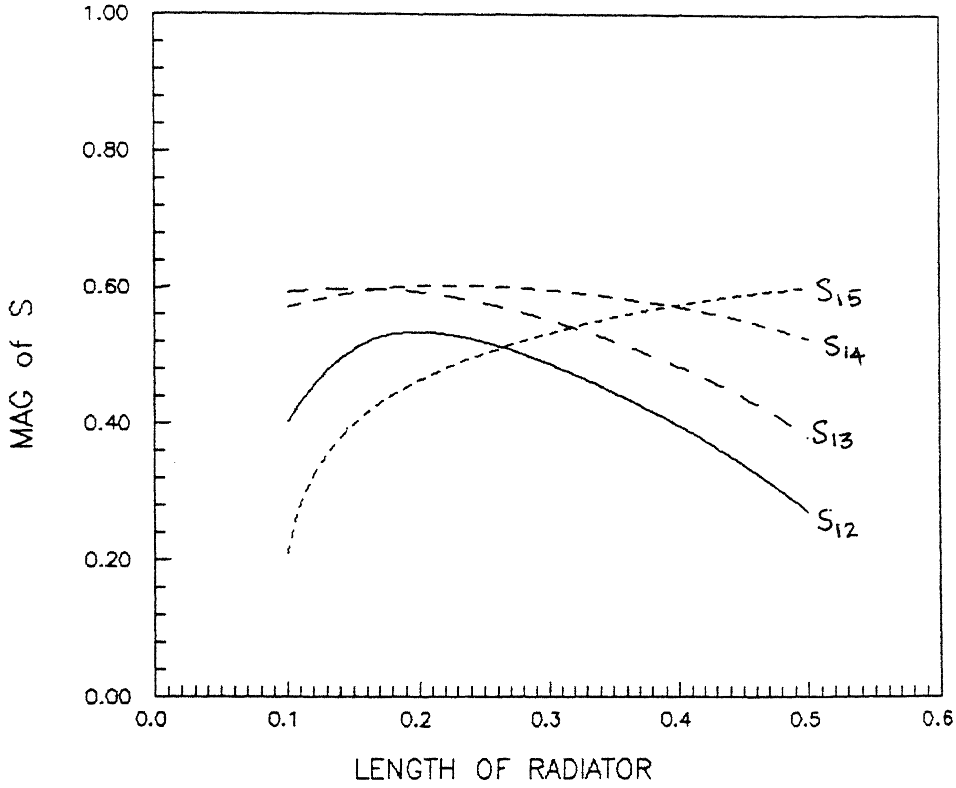


Fig 4.10 *Magnitude & angle of* S_{11} ,S_{22} ,S_{33} .



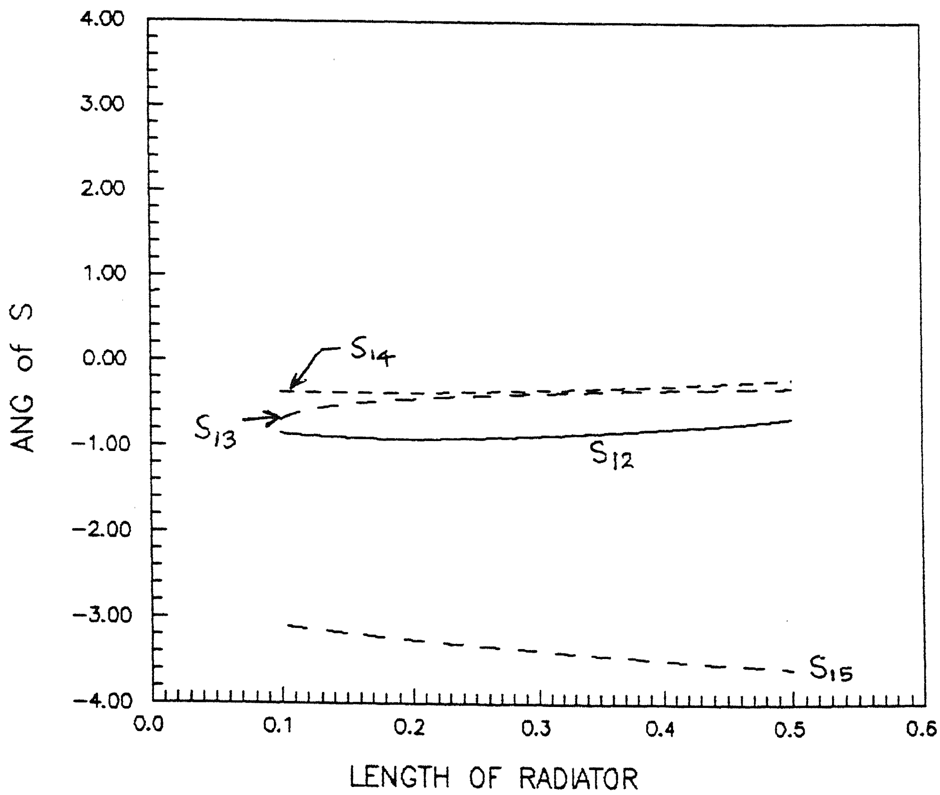
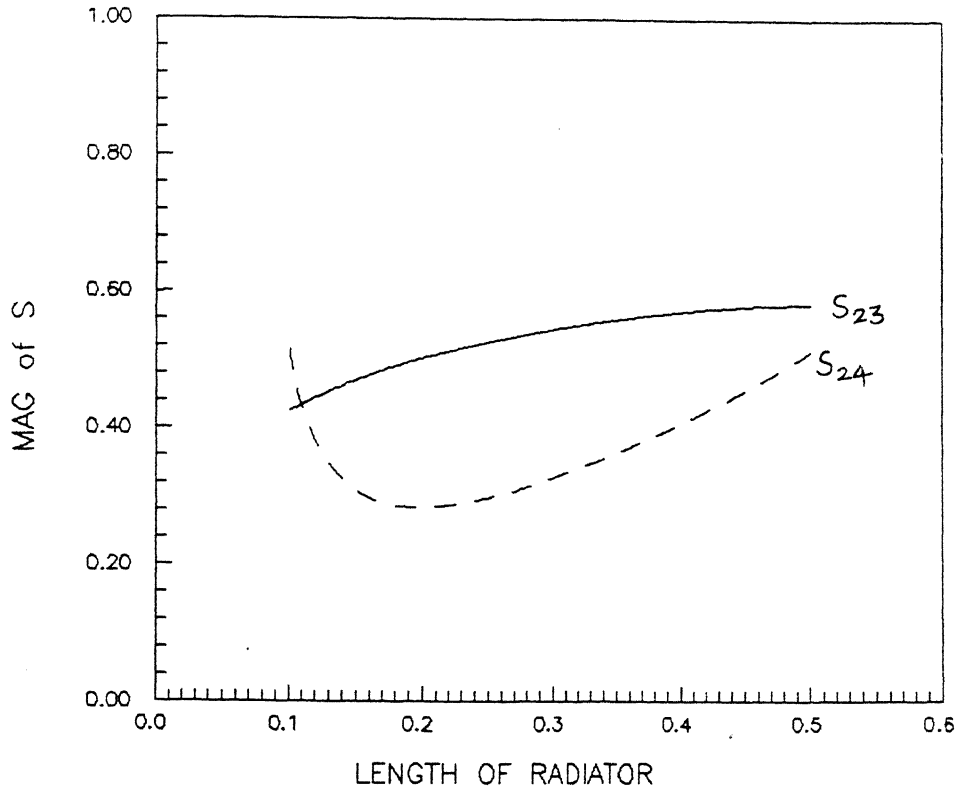


Fig 4.11 *Magnitude & angle of* S_{12} ,S_{13} ,S_{14} ,S_{15} .



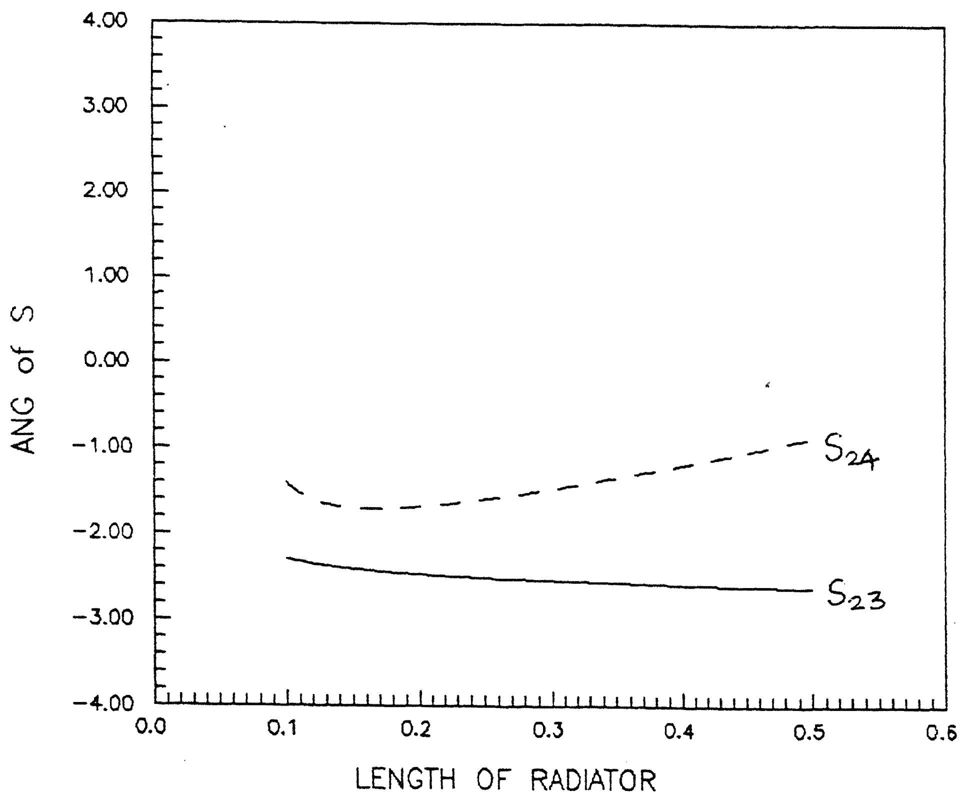


Fig 4.12 *Magnitude & angle of* S_{23} , S_{24}

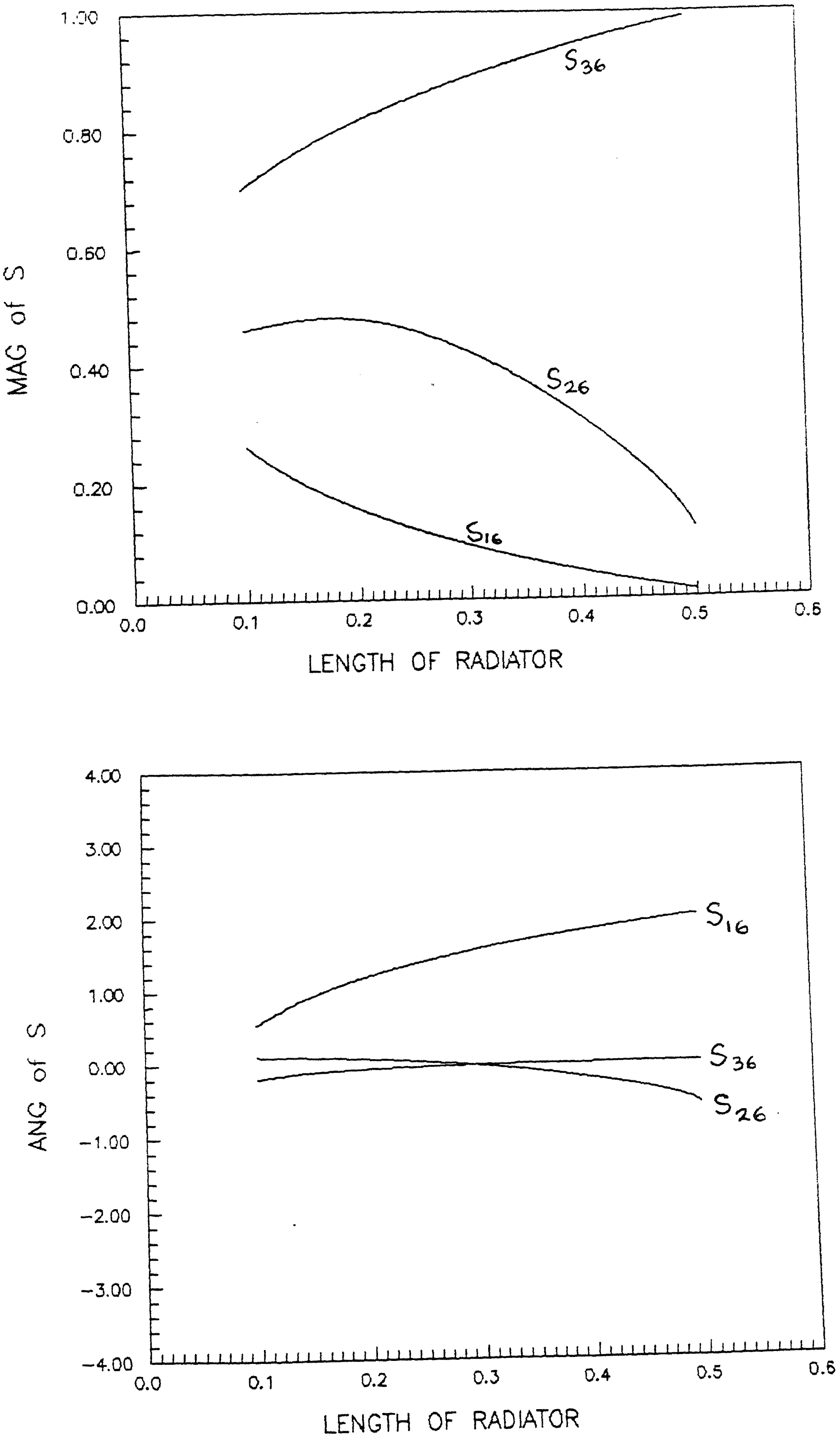


Fig 4.13 *Magnitude & angle of* S_{16} , S_{26} , S_{36} .

The optimisation process started with an assumed S-matrix for the feed~~ad~~ network at the lowest frequency of operation. then the feed network S-matrix is successfully updated to meet the requirements based on error criteria equation (4.33). The optimisation is carried out till the error is minimum possible. For the next higher frequency the optimisation is repeated to synthesize the feed network, with the initial guess of feed network S-parameters as that of immediate lower frequency. The results are presented in the form of graphs (4.10) to (4.12). The S-parameters have a gradual variation over the bandwidth. Such a smooth variation is desirable from the point of view of realization of the feed network. the reflection coefficient at the primary feed point is plotted in figure (4.1~~3~~4). It is observed that the source remains matched over the complete bandwidth. The reflection coefficient at the source port is maximum at the lowest operating frequency. This is due to the highly reactive nature of electrically small radiator. (which is very difficult to match as an individual radiator, as investigated in section 2.4). At the point when length becomes comparable to half wavelength, the reflection coefficient is very small conforming the characteristics and performance of convention radiator.

In order to test the performance of the array and feed network at the frequency within the bandwidth other than one at which it has been designed and optimesed, the s-parameters of feed network are interpolated at various points. The performance of the antenna are evaluated at these points and the reflection coefficient at the primary feed is found to be satisfactory. these points are also shown in figure (4.1~~3~~4).

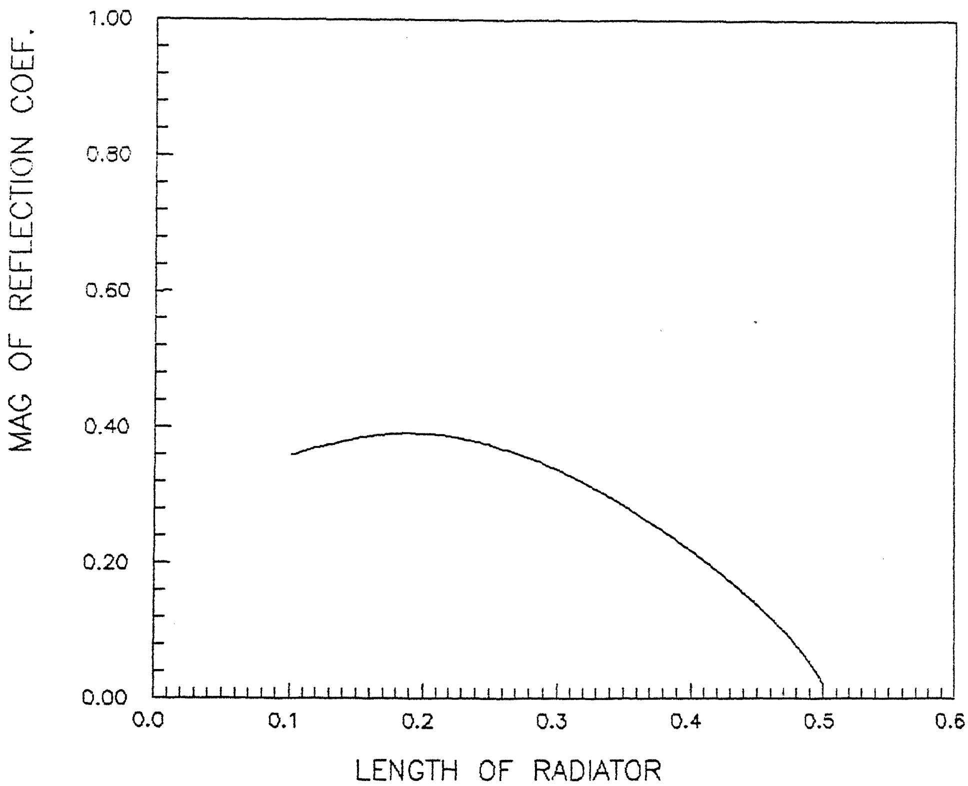


Fig. 4.14. Magnitude of Γ at feed point.

CHAPTER-5 RESULTS AND CONCLUSIONS

This chapter summarises the work done and the results reported in this thesis and also suggestions for further work.

This thesis presents a new configuration of antenna to meet the requirements of Direction Finding(DF) systems viz., wide band and constant beamwidth. The antenna is made up of electrically small radiators in colinear configuration, fed by a specially designed feed network. The feed network distributes the necessary excitations (as a function of frequency) in such a way that, the radiating currents flows over a constant electrical length throughout the bandwidth.

Individual electrically small radiator has been charecteerised as a function of frequency using the well known induced EMF method. The results show that electrically small radiator is inefficient, if operated as an individual antenna. Because the radiation resistance is much less compared to the reactance at the input terminals of the antenna. Next, mutual coupling between such electrically small radiators in colinear configuration is evaluated, to study the behaviour in array environment. In the colinear configuration, coupling is stronger between many adjacent elements at the lowest operating frequency and becomes significant only for immediate adjacent elements as the frequency increases.

An N-port network is analysed and a recursive relation is derived to evaluate the reflection coefficient at any port, when all the other (n-1) ports are loaded by a (n-1) port network. This analysis is used to synthesise the feed network terminal parameters. The scatter parameters of the feed network are arrived at using an optimisation method (pattern search), which provides minimum reflections at the feed point, and excites the required current distribution in the array. An array of five elements is designed to operate over multioctave bandwidth. The feed network S-parameters are also synthesised for the same; and the results are presented in the form of graphs. The synthesis is carried out at discrete points over the bandwidth. By interpolating the synthesised S-parameters of the feed network, the performance of the array is evaluated over the bandwidth. The results show a satisfactory input reflection coefficient over the entire bandwidth.

In this work dipoles are considered as electrically small radiators, but this topology can be applied to any other type of radiating element. The design of feed network and all the calculations were based on [S] or [Z] parameters so, the present work can be directly applied to any array, made up of electrically small radiators. Analytical formulation of self and mutual impedances may be difficult for complicated radiating geometry of individual elements in such cases, the measured scatter parameters can be used in the synthesis of fed network.

The present study was restricted to equal length of elements of the array. One may extend this work to optimise the performance of the array, by choosing unequal individual elements. The length of the elements could be chosen to have a gradual reduction in radiating length.

An important work will be to realise the feed network, characterized by scatter matrix. The synthesised terminal characteristics show a gradual variation as a function of frequency,and the feed network is symmetric about the center element. Hence it should be easy to realise the feed network. One may start with a conventional passive device like a T-junction or a cross (+) junction as a building block. The interconnection between these blocks and the characteristics of the block itself can be varied to achieve the terminal parameters of the feed network.

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APPENDIX 1

REFLECTION COEFFICIENT IN A GENERAL N-PORT NETWORK

Here the ratio of incident wave amplitudes (a_i/a_1) are derived for a general n-port network. These ratios are evaluated for 2-port, 3-port and 4-port networks, then by examining the expressions, a generalised formula is developed for n-port network.

case 1) Consider a two port network. The relationship between incident to reflected waves can be expressed as,

$$b_1 = S_{11}a_1 + S_{12}a_2 \qquad \ldots(a-1)$$

$$b_2 = S_{21}a_1 + S_{22}a_2 \qquad \ldots(a-2)$$

The reflection coefficient at port-1 (b_1/a_1) can be expressed as,

$$b_1/a_1 = S_{11} + S_{12}(a_2/a_1) \qquad \ldots(a-3)$$

Introducing the reflection coefficient of the load at port-2, $\Gamma_2 = (a_2/b_2)$ equation (a-2) becomes,

$$b_2/a_1 = S_{21} + S_{22}(a_2/a_1) \qquad \ldots(a-4)$$

$$(a_2/a_2)\, b_2/a_1 = S_{21} + S_{22}(a_2/a_1) \qquad \ldots(a-5)$$

$$(a_2/a_1)\,[1/\Gamma_2 - S_{22}] = S_{21} \qquad \ldots(a-6)$$

$$\boxed{\frac{a_2}{a_1} = \frac{S_{21}\,\Gamma_2}{1 - S_{22}\,\Gamma_2}} \qquad \ldots(a-7)$$

Similarly equation (a-3) can be expressed as

$$b_2/a_2 = S_{22} + S_{21}(a_1/a_2) \qquad \ldots(a-8)$$

where

$$\frac{a_1}{a_2} = \frac{S_{12}\,\Gamma_1}{1 - S_{11}\,\Gamma_1} \qquad \ldots(a-9)$$

case ii) Consider a three port network,

$$b_1 = S_{11}a_1 + S_{12}a_2 + S_{13}a_3 \qquad \ldots(a\text{-}10)$$

$$b_2 = S_{21}a_1 + S_{22}a_2 + S_{23}a_3 \qquad \ldots(a\text{-}11)$$

$$b_3 = S_{31}a_1 + S_{32}a_2 + S_{33}a_3 \qquad \ldots(a\text{-}12)$$

The reflection coefficient at port(1) is obtained from equation (a-10)

$$b_1/a_1 = S_{11} + S_{12}(a_2/a_1) + S_{13}(a_3/a_1) \qquad \ldots(a\text{-}13$$

where(a_2/a_1),(a_3/a_1) are obtained from equations (a-11) and (a-12) dividing equation (a-12) by a_1

$$b_3/a_1 = S_{31} + S_{32}(a_2/a_1) + S_{33}(a_3/a_1) \qquad \ldots(a\text{-}15)$$

substituting for $\Gamma_3 = a_3/b_3$ in equation (a-15) and bringing (a_3/a_1) terms together on LHS.

$$\frac{a_3}{a_1}\left[\frac{1}{\Gamma_3} - S_{33}\right] = S_{31} + S_{32}[a_2/a_1] \qquad \ldots(a\text{-}16)$$

$$\boxed{\frac{a_3}{a_1} = \frac{\Gamma_3}{1-\Gamma_3 S_{33}}\left[S_{31} + S_{32}[a_2/a_1]\right]} \qquad \ldots(a\text{-}17)$$

Fromequation(a-11)a_2/a_1)canbeobtainedinthefollowingmanner,

dividing equation (a-11) by a_1

$$b_2/a_1 = S_{21} + S_{22}(a_2/a_1) + S_{23}(a_3/a_1) \qquad \ldots(a\text{-}18)$$

substituting for $[a_3/a_1]$ from equation (a-17),

$$b_2/a_1 = S_{21} + S_{22}(a_2/a_1) + S_{23}\left\{\frac{\Gamma_3}{1-\Gamma_3 S_{33}}\left[S_{31} + S_{32}\frac{a_2}{a_1}\right]\right\} \ldots(a\text{-}19)$$

Expanding the term in curl. braces,

$$S_{21} + S_{22}(a_2/a_1) + \frac{\Gamma_3 S_{23} S_{31}}{1-\Gamma_3 S_{33}} + \frac{\Gamma_3 S_{32} S_{23}}{1-\Gamma_3 S_{33}}\left[\frac{a_2}{a_1}\right]$$

Substituting for Γ_2 and bringing the (a_2/a_1) terms to the LHS.

$$\left[\frac{a_2}{a_1}\right]\left\{\frac{1}{\Gamma_2} - S_{22} - \frac{\Gamma_3 S_{32} S_{23}}{1-\Gamma_3 S_{33}}\right\} = S_{21} + \frac{\Gamma_3 S_{31} S_{23}}{1-\Gamma_3 S_{33}} \qquad \ldots(A\text{-}20)$$

$$\boxed{\left[\frac{a_2}{a_1}\right] = \frac{\Gamma_2 \left\{ S_{21} + \dfrac{\Gamma_3 S_{31} S_{23}}{1 - \Gamma_3 S_{33}} \right\}}{1 - \Gamma_2 \left\{ S_{22} + \dfrac{\Gamma_3 S_{32} S_{23}}{1 - \Gamma_3 S_{33}} \right\}}} \quad \text{...(a-21)}$$

case iii) Consider a four port network,

$$b_1 = S_{11} a_1 + S_{12} a_2 + S_{13} a_3 + S_{14} a_4 \quad \text{...(a-22)}$$

$$b_2 = S_{21} a_1 + S_{22} a_2 + S_{23} a_3 + S_{24} a_4 \quad \text{...(a-23)}$$

$$b_3 = S_{31} a_1 + S_{32} a_2 + S_{33} a_3 + S_{34} a_4 \quad \text{...(a-24)}$$

$$b_4 = S_{41} a_1 + S_{42} a_2 + S_{43} a_3 + S_{44} a_4 \quad \text{...(a-25)}$$

The reflection coefficient at port(1) can be expressed as,

$$b_1/a_1 = S_{11} + S_{12}(a_2/a_1) + S_{13}(a_3/a_1) + S_{14}(a_4/a_1) \quad \text{...(a-26)}$$

From equation (a-25)

$$b_4/a_1 = S_{41} + S_{42}(a_2/a_1) + S_{43}(a_3/a_1) + S_{44}(a_4/a_1) \quad \text{...(a-27)}$$

Replacing b_4/a_1 by $(a_4/a_1)/\Gamma_4$ and collecting the terms containing (a_4/a_1) to Left Hand Side, equation (a-27) becomes,

$$\frac{a_4}{a}\left[\frac{1}{\Gamma} - S_{44}\right] = S_{41} + S_{42}[a_2/a_1] + S_{43}[a_3/a_1]$$

$$\boxed{\frac{a_4}{a_1} = \frac{\Gamma_4}{1-\Gamma_4 S_{44}}\left[S_{41} + S_{42}[a_2/a_1] + S_{43}[a_3/a_1]\right]} \quad \text{...(a-28)}$$

Dividing equation (a-26) by a_1,

$$b_3/a_1 = S_{31} + S_{32}(a_2/a_1) + S_{33}(a_3/a_1) + S_{34}(a_4/a_1) \quad \text{...(a-29)}$$

Substituting for (a_4/a_1) from equation (a-28),

$$b_3/a_1 = S_{31} + S_{32}(a_2/a_1) + S_{33}(a_3/a_1) +$$

$$S_{34}\left\{ \frac{\Gamma_4}{1 - \Gamma_4 S_{44}} \left[S_{41} + S_{42}[a_2/a_1] + S_{43}[a_3/a_1] \right] \right\} \quad \text{...(a-30)}$$

Replacing b_3/a_1 by $(a_3/a_1)/\Gamma_3$ and collecting the terms containing (a_3/a_1) to Left Hand Side, equation (a-30) becomes,

$$\left[\frac{a_3}{a_1}\right]\left\{\frac{1}{\Gamma_3}-S_{33}-\frac{\Gamma_4\,S_{43}\,S_{34}}{1-\Gamma_4\,S_{44}}\right\}=S_{31}+\frac{\Gamma_4 S_{41} S_{34}}{1-\Gamma_4 S_{44}}+\left[\frac{a_2}{a_1}\right]\left\{S_{32}+\frac{\Gamma_4\,S_{42}\,S_{34}}{1-\Gamma_4\,S_{44}}\right\}\qquad\ldots(a\text{-}31)$$

$$\left[\frac{a_3}{a_1}\right]=\frac{\Gamma_3\left\{S_{31}+\dfrac{\Gamma_4\,S_{41}\,S_{34}}{1-\Gamma_4\,S_{44}}+\left[\dfrac{a_2}{a_1}\right]\left\{S_{32}+\dfrac{\Gamma_4\,S_{42}\,S_{34}}{1-\Gamma_4\,S_{44}}\right\}\right\}}{1-\Gamma_3\left\{S_{33}+\dfrac{\Gamma_4\,S_{43}\,S_{34}}{1-\Gamma_4\,S_{44}}\right\}}\qquad\ldots(a\text{-}32)$$

Now consider the equation (a-23),

$$b_2/a_1=S_{21}+S_{22}(a_2/a_1)+S_{23}(a_3/a_1)+S_{24}(a_4/a_1)\qquad\ldots(a\text{-}33)$$

$$b_2/a_1-S_{22}(a_2/a_1)=S_{21}+S_{23}(a_3/a_1)+S_{24}(a_4/a_1)\qquad\ldots(a\text{-}34)$$

Substituting for (b_2/a_1) LHS becomes,

$$\frac{a_2}{a_1}\left[\frac{1}{\Gamma_2}-S_{22}\right]=\left[\frac{a_2}{a_1}\right]\frac{1-\Gamma_2\,S_{22}}{\Gamma_2}\qquad\ldots(a\text{-}35)$$

In RHS of equation (a-34) (a_4/a_1) is replaced by the respective expression (a-28), to yield:

$$S_{21}+\frac{\Gamma_4 S_{41} S_{24}}{1-\Gamma_4 S_{44}}+S_{23}\left[\frac{a_3}{a_1}\right]$$

$$\left[\frac{a_3}{a_1}\right]\left\{\frac{\Gamma_4\,S_{43}\,S_{24}}{1-\Gamma_4\,S_{44}}\right\}+\left[\frac{a_2}{a_1}\right]\left\{\frac{\Gamma_4\,S_{42}\,S_{24}}{1-\Gamma_4\,S_{44}}\right\}\qquad\ldots(a\text{-}36)$$

Bringing the last term in the above equation to LHS of (a-34)

$$\frac{a_2}{a_1}\left(\frac{1}{\Gamma_2}-S_{22}-\frac{\Gamma_4 S_{24} S_{42}}{1-\Gamma_4 S_{44}}\right)=S_{21}+\frac{\Gamma_4 S_{41} S_{24}}{1-\Gamma_4 S_{44}}+\frac{a_3}{a_1}\left\{\frac{\Gamma_4\ S_{43}\ S_{24}}{1-\Gamma_4\ S_{44}}+S_{23}\right\} \quad ..(a37)$$

Consider the RHS of equation (a-37),replacing (a_3/a_1) by the expression derived in equation (a-32),using the short hand notations A,B,C,D defined below RHS becomes,

$$A+B\ \frac{\left\{\Gamma_3\ [D+E\left[\frac{a_2}{a_1}\right]]\right\}}{C} \qquad ...(a-38)$$

Where,

$$A=S_{21}+\frac{\Gamma_4 S_{41} S_{24}}{1-\Gamma_4 S_{44}}\ ; \qquad B=\frac{\Gamma_4\ S_{43}\ S_{24}}{1-\Gamma_4\ S_{44}}+S_{23}$$

$$C=1-\Gamma_3\left\{S_{33}+\frac{\Gamma_4\ S_{43}\ S_{34}}{1-\Gamma_4\ S_{44}}\right\} \qquad D=S_{31}+\frac{\Gamma_4\ S_{41}\ S_{34}}{1-\Gamma_4\ S_{44}}$$

And $E = S_{32}+(S_{34}S_{42}\Gamma_4\ /(1-\Gamma_4 S_{44}))$

Equating (a-38) to LHS of (a-31) we get the numerator of (a_2/a_1) as

$$\Gamma_2[\frac{\Gamma_3\left\{S_{23}+\frac{\Gamma_4\ S_{43}\ S_{24}}{1-\Gamma_4\ S_{44}}\right\}\left\{S_{31}+\frac{\Gamma_4\ S_{41}\ S_{34}}{1-\Gamma_4\ S_{44}}\right\}}{1-\Gamma_3\left\{S_{33}+\frac{\Gamma_4\ S_{43}\ S_{34}}{1-\Gamma_4\ S_{44}}\right\}}+\left\{S_{21}+\frac{\Gamma_4\ S_{24}\ S_{41}}{1-\Gamma_4\ S_{44}}\right\}]$$

and the denominator as,

$$1-\Gamma_2[\frac{\Gamma_3\left\{S_{23}+\frac{\Gamma_4\ S_{43}\ S_{24}}{1-\Gamma_4\ S_{44}}\right\}\left\{S_{32}+\frac{\Gamma_4\ S_{42}\ S_{42}}{1-\Gamma_4\ S_{44}}\right\}}{1-\Gamma_3\left\{S_{33}+\frac{\Gamma_4\ S_{43}\ S_{34}}{1-\Gamma_4\ S_{44}}\right\}}+\left\{S_{22}+\frac{\Gamma_4\ S_{24}\ S_{42}}{1-\Gamma_4\ S_{44}}\right\}]$$

This completes the required formula for a four port network.
By comparing the equations of (a_2/a_1) of two port network from (a-7) with that of 3 port network equation(a-21)

$$^{2}\left(\frac{a_2}{a_1}\right) = \frac{S_{21}\ \Gamma_2}{1 - S_{22}\ \Gamma_2} \qquad \text{...(a-7)}$$

Inclusion of third port leads to changing S_{21} to S_{21}+ $(S_{31}S_{23}\Gamma_3/(1-S_{33}\Gamma_3))$and $S_{22}$to S_{22}+ $(S_{32}S_{23}\Gamma_3/(1-S_{33}\Gamma_3))$. That is replacing S_{ij}in (n-1) port network to S_{ij}+ $(S_{nj}S_{in}\Gamma_n/(1-S_{nn}\Gamma_n))$ gives the expression for the n-port network.

$$^{3}\left(\frac{a_2}{a_1}\right) = \frac{\Gamma_2 \left\{ S_{21} + \dfrac{\Gamma_3\ S_{31}\ S_{23}}{1 - \Gamma_3\ S_{33}} \right\}}{1 - \Gamma_2 \left\{ S_{22} + \dfrac{\Gamma_3\ S_{32}\ S_{23}}{1 - \Gamma_3\ S_{33}} \right\}} \qquad \text{...(a-21)}$$

Replacing likewise for a four port network leads to the same formula derived equation (a-39).Hence the formula for a general n-port network can be written based on the observations.

$$\left(\frac{a_2}{a_1}\right) = \frac{\Gamma_2\ [S_{21}]}{1 - \Gamma_2\ [S_{22}]}$$

The term inside the square bracket builds up n-times according to the S_{ij}expression mentioned in the above paragraph., where n is the number of ports. This build up can be expressed by a mutual recursive formula(meaning the same expression is substituted for each variable expression repeatedly).

$$^{n}\left(\frac{a_2}{a_1}\right) = \frac{\Gamma_2\ [S_{21}]_2}{1 - \Gamma_2\ [S_{22}]_2} \qquad \text{...(a-40)}$$

where

$$\left[S_{ij}\right]_x = \frac{\left[S_{i,x+1}\right]_{x+1}\ \left[S_{x+1,j}\right]_{x+1}\ \Gamma_{x+1}}{1 - \Gamma_{x+1}\ \left[S_{x+1,x+1}\right]_{x+1}} + \left[S_{i,j}\right]_{x+1} \qquad \text{...(a-41)}$$

$$\left[S_{ij}\right]_n = S_{ij} \qquad \ldots(a-42)$$

Using (a-40) (a_2/a_1) of a general network can be computed. Applying the same logic to other ratios (a_3/a_1)... one can calculate them if the same ratio for a network of a lower order is known. We shall find out a general formula to represent the ratio (a_m/a_1) of an n-port network. Consider the case of (a_n/a_1) of two, three and four port networks.

case i) n=2,

$$\left[\frac{a_n}{a_1}\right] = \frac{\Gamma_2 \; S_{21}}{1 - \Gamma_2 \; S_{22}} \qquad (a-43)$$

case i) n=3,

$$\left[\frac{a_n}{a_1}\right] = \frac{\Gamma_3}{1 - \Gamma_3 \; S_{33}} \left\{ S_{31} + S_{32} \left[\frac{a_2}{a_1}\right] \right\} \qquad \ldots(a-44)$$

Replacing all Γ_n to Γ_{n+1}, $S_{n,n}$ to $S_{n+1,n+1}$ and $S_{n,1}$ by

$$S_{n,1} = S_{n+1,1} + \sum_{i=2}^{n} S_{n+1,i} \left[\frac{a_i}{a_1}\right] \qquad \ldots(a-45)$$

Hence for n=4 we get

$$\frac{a_4}{a_1} = \frac{\Gamma_4}{1 - \Gamma_4 S_{44}} \left[S_{41} + S_{42}[a_2/a_1] + S_{43}[a_3/a_1]\right]$$

Which is same as equation (a-28)

Generalised expression can be obtained by combining the above expression with (a-40). The ratio (a_m/a_1) of n-port network can be written as,

$$^{n}\left[\frac{a_m}{a}\right] = \frac{\Gamma_m}{1 - \Gamma_m \left(S_{mm}\right)_m} \sum_{q=1}^{m-1} \left\{ \left(S_{mq}\right)_m \left[\frac{a_q}{a_1}\right] \right\} \qquad \ldots(a-46)$$

So for we have generalised the incident wave ratios at any port m with respect to incident wave at port 1 in an n-port network. But our interest is to calculate the reflection coefficient at any port l. It will be required to evaluate (a_m/a_l) of n-port network. Direct evaluation of a formula for this is complicated. As this evaluated by a computer, one can simply manipulate the matrix of Γ and S such a way that the l^{th} port is shifted to the position of port(1) and evaluate (a-46). After the evaluation care should be taken to re-manipulate the S matrix and Γ matrix to yield the original set.